Solution of Ordinary Differential Equations

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Chapter 1

Introduction

1.1 Basic Concepts and Definitions

Differential Equations: The laws of the universe are written in the language of mathematics. Algebra is sufficient to solve many static problems, but the most interesting natural phenomena involve change and are described by equations that relate changing quantities. Because the derivative $f'(t)$ of the function f is the rate at which the quantity $f(t)$ is changing with respect to the independent variable[1] t, it is natural that equations involving derivatives are frequently used to describe the changing universe. An equation relating an unknown function and one or more of its derivatives is called a differential equation, (DE).

A DE is used to describe changing quantities and it plays a major role in qualitative studies in many disciplines such as all areas of engineering, physical sciences, life sciences, and economics.

The differential equation

$$\frac{dx}{dt} = x^2 + t^2$$

involves both the unknown function $x(t)$ and its first derivative $x'(t) = \frac{dx}{dt}$.
The differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 7y = 0$$

involves the unknown function y of the independent variable x and the first two derivatives $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ of y.

Examples:
Are they DEs or not?
$ax^2 + bx + c = 0$ No!
$ax^2 + bx' + c = 0$ Yes! Here $x' = \frac{dx}{dt}$

[1]When an equation involves one or more derivatives with respect to a particular variable,that variable is called an independent variable. A variable is called dependent if a derivative of that variable is occurs.

$ax^2 + bx^{'} + cy^{'} = 0$ Yes! Here $x^{'} = \frac{dx}{dt}$ and $y^{'} = \frac{dy}{dx}$
$y^{''} = x^3$ Yes! Here $y^{''} = \frac{d^2y}{dx^2}$.

Applications of Differential Equations:
As mentioned before, in real life there are a lot of applications of DEs such as e.g. Newton's law of cooling, where the physical law can be translated into a differential equation. Another examples are Torricelli's law, heat transfer in materials, relaxation in nuclear magnetic resonance, radioactive decay, and chemical reaction kinetics where the rate law or rate equation for a chemical reaction is a DE that links the reaction rate with concentrations or pressures of reactants and constant parameters.

1.2 Classification of Differential Equation

There are some ways of classifying differential equations.

1. Order and Degree of DEs: The order of a DE is determined by the highest order derivative of the dependent variable.

   Examples: Determine the order of the following DEs:

   $$ax + \frac{dx}{dt} = 0 \qquad \text{"first order DE"}$$

   $$\frac{dx}{dt} + ax^2 = 0 \qquad \text{"first order DE"}$$

   $$\frac{d^2x}{dt^2} + bx = 0 \qquad \text{"second order DE"}$$

   $$bx^4 + \frac{d^2x}{dt^2} = 0 \qquad \text{"second order DE"}$$

   $$\frac{d^{(n)}x}{dt^{(n)}} + cx = 0 \qquad \text{"n} - \text{th order DE".}$$

   While the degree of a DE is determined by the highest power of the highest order derivative present in the equation.
   Examples:

   $$(\frac{d^2x}{dt^2})^5 + b(\frac{d^3x}{dt^3})^2 = 0 \qquad \text{"second degree third order DE"}$$

   $$\frac{dx}{dt} + ax^3 = 0 \qquad \text{"first degree first order DE"}$$

   $$(\frac{d^2x}{dt^2})^{(n)} + a\frac{dx}{dt} + bx = 0 \qquad \text{"n} - \text{th degree second order DE".}$$

2. Homogeneous and Nonhomogeneous DEs: A differential equation is nonhomogeneous if it has terms involving only the independent variable (and constants) on the right hand side, and it is homogeneous if this right hand side is zero.
Examples: The differential equations

$$\frac{dy}{dx} + x^2 y = 4x^3$$

$$\frac{d^4y}{dx^4} + x\frac{d^2y}{dx^2} + y^2 = 6x + 3$$

are nonhomogeneous.
While the following differential equation

$$\frac{d^4y}{dx^4} + x\frac{d^2y}{dx^2} + y^2 = 0.$$

is homogeneous.

3. Linear and Nonlinear DEs: A crucial classification of differential equations is whether they are linear or nonlinear. The differential equation is said to be linear if it is a linear function in the dependent variable y and its derivatives, but not in x.

Examples: The differential equation

$$e^x\frac{d^2y}{dx^2} + (cosx)\frac{dy}{dx} + (1 - \sqrt{x})y = tan^{-1}x$$

is linear because the dependent variable y and its derivatives $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ appear linearly. By contrast, the equations

$$\frac{d^2y}{dx^2} = y\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} + 3(\frac{dy}{dx})^2 + 4y^3 = 0$$

are not linear because products and powers of y or its derivatives appear. Also,

$$\frac{d^2y}{dx^2} + 2(\frac{dy}{dx})^3 + y = 0 \qquad \text{is nonlinear}$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + x^2y = 4x^3 \qquad \text{is linear}$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0 \qquad \text{is linear.}$$

4. Ordinary and Partial Differential Equations: One of the more obvious classifications is based on whether the unknown function depends on a single independent variable or on several independent variables. In the first case, only ordinary derivatives appear in the differential equation, and it is said to be an *Ordinary Differential Equation*, (ODE). In the second case, the derivatives are partial derivatives, and the equation is called a *Partial Differential Equation* (PDE).

   Examples: All the above differential equations, which were discussed before, are ODEs. Another example of ODE is

$$\frac{d^2x(t)}{dt^2} + \frac{dx(t)}{dt} + x(t) = 0 \qquad \text{"ODE"}$$

   While examples of PDEs are

$$\frac{\partial^2 y(t,s)}{\partial s^2} - \frac{\partial y(t,s)}{\partial t} = 0 \qquad \text{"PDE"}$$

$$\frac{\partial^2 y(t,s)}{\partial s^2} - \frac{\partial^2 y(t,s)}{\partial t^2} = 0 \qquad \text{"PDE"}$$

   In this course, only ODEs are considered.

1.3 Solutions of ODE

In algebra, we typically seek the unknown numbers that satisfy an equation such as $x^3 + 7x^2 - 11x + 41 = 0$. By contrast, in solving a differential equation, we are challenged to find the unknown function $y = y(x)$ for which an identity such as

$$\frac{dy}{dx} = 2xy$$

holds on some interval of real numbers. Ordinarily, we will want to find all solutions of the differential equation, if possible.
Example:

$$ay'(x) + b = 0$$

$$y' = -\frac{b}{a} \qquad \text{is not a solution}$$

$$y = -\int \frac{b}{a}\, dx \qquad \text{is a solution.}$$

Example: Show that $y = e^{2x}$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0.$$

Solution:

$$y = e^{2x} \quad \Longrightarrow \frac{dy}{dx} = 2\ e^{2x} \Longrightarrow \frac{d^2y}{dx2} = 4\ e^{2x};$$

then

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 4\ e^{2x} + 2\ e^{2x} - 6\ e^{2x} = 0.$$

There are some methods for finding the solution of the differential equations, these methods are discussed in the following next chapters.

Chapter 2

Methods of Finding the Solution of the First Order Differential Equation

The first-order DE of degree one

$$F(x, y, \frac{dy}{dx}) = 0,$$

could be written in one of the following forms

$$\frac{dy}{dx} = f(x, y)$$

$$y' = f(x, y)$$

$$M(x, y)dx + N(x, y)dy = 0.$$

For examples: The following are first-order DE's

1. $y' = 2xy$
2. $(x^2 + y^2)dx + (x^2 - 2xy + y^2)dy = 0$
3. $\frac{dy}{dx} = \frac{x - y + 1}{2x + 3y - 4}$
4. $\frac{dy}{dx} - 2xy = e^x cosx$
5. $3x(xy - 2)dx + (x^3 + 2y)dy = 0$
6. $y(6y^2 - x - 1)dx + 6y^3dy = 0.$

There are some methods of finding the solution of the first order DE based on integration as general and particular solutions.

2.1 Separation of Variables

Given the first order diff. eq.

$$F(x, y, \frac{dy}{dx}) = 0 \tag{2.1}$$

If eq. (2.1) can be written as

$$\frac{dy}{dx} = f(x)g(y) \tag{2.2}$$

where $f(x)$ is a function of x alone; and $g(y)$ is a function of y alone,
then the variables of (2.2) can be separated as

$$\frac{dy}{g(y)} = f(x)dx \tag{2.3}$$

and the general solution can be found by integrating both sides of (2.3), i.e.

$$\int \frac{dy}{g(y)} = \int f(x)dx + c. \tag{2.4}$$

Example 1. Find the solution of

$$\frac{dy}{dx} = \frac{x^2}{1-y^2} \tag{2.5}$$

Solution: The given eq. in (2.5) is separable. Rewrite it as

$$(1-y^2)\,dy = x^2\,dx.$$

In order to solve the above equation, we have to integrate the left hand side with respect to y while the right hand side with respect to x

$$\int (1-y^2)\,dy = \int x^2\,dx$$

$$y - \frac{y^3}{3} = \frac{x^3}{3} + c$$

$$3y - y^3 = x^3 + 3c \tag{2.6}$$

where c is an arbitrary constant.

Note: In eq.(2.5), if in addition to the differential equation, an initial condition

$$y(x_0) = y_0$$

is prescribed, then the solution of eq.(2.5) satisfying this condition is obtained by setting $x = x_0$ and $y = y_0$ in eq.(2.6). This gives the particular value of c in eq. (2.6).
In this case the solution is known as the particular solution rather that the general solution.

Example 2. Solve the following initial value problem

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad y(0) = -1.$$

Solution: The differential equation can be written as

$$2(y-1)dy = (3x^2 + 4x + 2)dx.$$

Integrate the left hand side with respect to y and the right hand side with respect to x gives

$$\int 2(y-1)dy = \int (3x^2 + 4x + 2)dx$$

$$y^2 - 2y = x^3 + 2x^2 + 2x + c \tag{2.7}$$

where c is an arbitrary constant.
In order to determine the value of c in eq.(2.7), we substitute the given initial condition $(y(0) = -1)$ in eq.(2.7), obtaining $c = 3$.
Hence, the particular solution of the initial value problem is

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3.$$

Example 3. Solve the initial value problem

$$\frac{dy}{dx} = -6xy, \quad y(0) = 7.$$

Solution: Divide both sides of the DE by y and multiply each side by dx to get

$$\frac{dy}{y} = -6x\ dx. \tag{2.8}$$

Then integrate both sides

$$\int \frac{dy}{y} = \int (-6x)\ dx$$

$$ln\ y = -3x^2 + c$$

$$y = e^{-3x^2+c} = e^{-3x^2}e^c \tag{2.9}$$

let

$$e^c = A$$

then eq.(2.9) is redefined as

$$y = Ae^{-3x^2} \tag{2.10}$$

substitute the initial condition $(y(0) = 7)$ in eq.(2.10) yields $A = 7$.
So, the particular solution is

$$y = 7e^{-3x^2}.$$

Example 4.
Find the solution of the initial value problem

$$\frac{dy}{dx} = \frac{y\ cosx}{1+2y^2}, \qquad y(0) = 1$$

Solution: Rewrite the DE as $\dfrac{1+2y^2}{y}\ dy = cosx\ dx$, such that the resulting equation is separable. Hence

$$\int \frac{1+2y^2}{y}\ dy = \int cosx\ dx$$

$$ln\ y + y^2 = sinx + c.$$

Substitute the given initial condition, gives $c = 1$. Hence, the solution of the DE is

$$ln\ y + y^2 = sinx + 1.$$

Example 5. Solve

$$(1+e^y)\ dx - csc^2x\ dy = 0.$$

Solution:

$$csc^2x\ dy = (1+e^y)\ dx \implies \frac{dy}{1+e^y} = \frac{dx}{csc^2x}$$

$$\int \frac{dy}{1+e^y} = \int sin^2x\ dx$$

$$\int \frac{e^{-y}\ dy}{e^{-y}+1} = \int \frac{1-cos2x}{2} dx$$

$$-ln(e^{-y}+1) = \frac{1}{2}\ x - \frac{1}{4}\ sin2x + c.$$

Application in Reality

Cooling and Heating Equation:
Newton's law of cooling may be stated in this way:
The time rate of change of the temperature $T(t)$ of a body is proportional to the difference between T and the temperature A of the surrounding medium. That is

$$\frac{dT}{dt} = k(A-T) \tag{2.11}$$

where k is a positive constant. Observe that if $T > A$, then $\frac{dT}{dt} < 0$, so the temperature is a decreasing function of t and the body is cooling. But if $T < A$, then $\frac{dT}{dt} > 0$, so that T is increasing.
Thus the physical law is translated into a differential equation. If we are given the values

of k and A, we should be able to find an explicit formula for $T(t)$, and then –with the aid of this formula– we can predict the future temperature of the body.

Example 5.
A roast, initially at 50F, is placed in a 375F oven at 5:00 Pm. After 75 minutes it is found that the temperature $T(t)$ of the roast is 125F. When will the roast be 150F (medium rare)?

Solution: We take t in minutes, with $t = 0$ corresponding to 5:00 pm.
We have $T(t) < A = 375$, $T(0) = 50$, and $T(75) = 125$. Hence

$$\frac{dT}{dt} = k(375 - T)$$
$$\int \frac{dT}{375 - T} = \int k\, dt$$
$$-ln(375 - T) = kt + c$$
$$375 - T = Be^{-kt}.$$

Substitute the initial condition $T(0) = 50$,

$$\implies 375 - 50 = Be^0 \implies B = 325,$$

so

$$375 - T = 325e^{-kt},$$

or

$$T(t) = 375 - 325\, e^{-kt}. \tag{2.12}$$

We also know that $T(75) = 125$, substitute in (2.12) implies

$$125 = 375 - 325\, e^{(-75)k} \implies 250 = 325\, e^{(-75)k}$$
$$\implies ln(\frac{250}{325}) = -75k$$

which yields

$$k = -\frac{1}{75}\, ln(\frac{250}{325}) \approx 0.0035.$$

Substitute the value of k and $T(t) = 150$ in (2.12), and hence we have to solve the equation

$$150 = 375 - 325e^{(-0.0035)t}$$

for $t = -\dfrac{ln(225/325)}{0.0035} \approx 105$ min, the total cooking time required. Because the roast was placed in the oven at 5:00 pm, it should be removed at about 6:45 pm.

Exercises:

A. Find the particular solution of

1. $2xy\,\dfrac{dy}{dx} = 1 + y^2, \qquad y(2) = 3.$
2. $xy^2dx + e^x dy = 0, \qquad y(0) = \dfrac{1}{2}.$

B. Find the general solution of the following

1. $(1 - x)\dfrac{dy}{dx} = y^2.$
2. $sinx\ cosy\ dx + cosx\ siny\ dy = 0.$
3. $ye^{2x}\ dx = (4 + e^{2x})\ dy.$

2.2 Homogeneous Equations (Equations Transformed to Separable Equations)

Homogeneous functions: The function $f(x,y)$ is said to be homogeneous of degree k in x and y if, and only if,

$$f(\lambda x, \lambda y) = \lambda^k f(x,y).$$

The definition is easily extended to functions of more than two variables.

Examples:

1. $f(x,y) = 2y^3\ exp(\frac{y}{x}) - \dfrac{x^4}{x+3y}$

$f(\lambda x, \lambda y) = 2\lambda^3 y^3\ exp(\frac{\lambda y}{\lambda x}) - \dfrac{\lambda^4 x^4}{\lambda x + 3\lambda y} = \lambda^3 f(x,y);$
hence $f(x,y)$ is homogeneous of degree 3 in x and y.

2. $f(x,y) = \sqrt{x+4y}$

$f(\lambda x, \lambda y) = \sqrt{\lambda x + 4\lambda y} = \lambda^{\frac{1}{2}}\ f(x,y);$
hence the function $f(x,y) = \sqrt{x+4y}$ is homogeneous of degree $\dfrac{1}{2}$ in x and y.

3. $f(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}$

$f(\lambda x, \lambda y) = \dfrac{\lambda x}{\sqrt{\lambda^2 x^2 + \lambda^2 y^2}} = \lambda^0 f(x,y);$
hence the function $f(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}$ is homogeneous of degree zero in x and y.

Definition: The first-order ODE

$$M(x,y)dx + N(x,y)dy = 0. \tag{2.13}$$

is said to be homogeneous if $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same degree.

Remark: In order to solve the homogeneous differential equation, we let $y = vx \implies dy = vdx + xdv$ and substituting in the homogeneous diff. eq. (2.13) which will be reduces to a separable equation.
After finding the solution, we have to back to the original variables by inserting $\dfrac{y}{x}$ for v.

Solution steps:
If M and N in (2.13) are homogeneous functions of the same degree, then follow these steps:

1. Substitute $y = vx$, $dy = vdx + xdv$, then expand all the brackets and simplify
2. Collect the variables that corresponding to dx or dv
3. Extract and divide by a common factor in order to convert the equation to a separable one
4. After solving the separable equation, back to the original variables via re-substitution.

Remark: The above method would also be successful if we used $x = vy$ (instead of $y = vx$). However, it is sometimes easier to substitute for the variable whose differential has the simpler coefficient.

Example 1. Solve the equation

$$(x^2 - xy + y^2)\, dx - xy\, dy = 0. \tag{2.14}$$

Solution: Since the coefficients in (2.14) are both homogeneous and of degree two in x and y, let us put $y = vx$, $\quad dy = v\, dx + x\, dv$.
Then (2.14) becomes

$$(\quad (x^2 - x^2v + x^2v^2)\, dx - x^2v(v\, dx + x\, dv) = 0, \quad) * \frac{1}{x^2}$$

$$(1 - v + v^2)\, dx - v(v\, dx + x\, dv) = 0$$

$$(1 - v)\, dx + v^2\, dx - v^2\, dx - vx\, dv = 0.$$

$$(\quad (1 - v)\, dx - xv\, dv = 0. \quad) * \frac{-1}{x(v - 1)}$$

Hence we separate variables to get

$$\frac{dx}{x} + \frac{v\, dv}{v - 1} = 0$$

$$\int \frac{dx}{x} + \int (1 + \frac{1}{v - 1})dv = 0$$

$$ln\ x + v + ln(v - 1) = ln\ c$$

or

$$x(v - 1)\, e^v = c.$$

In terms of the original variables, the solution is given by

$$x(\frac{y}{x} - 1)\, exp(\frac{y}{x}) = c$$

or

$$(y - x)\, exp(\frac{y}{x}) = c.$$

Example 2. Solve the equation

$$y\,dx = (x + \sqrt{y^2 - x^2})\,dy. \tag{2.15}$$

Solution: The coefficients in the equation are homogeneous and of degree one.
We could use $y = vx$, but the relation simplicity of the dx term in (2.15) suggests that we put $x = vy$. Then $dx = v\,dy + y\,dv$, and equation (2.15) is replaced by

$$y\,(v\,dy + y\,dv) = (vy + \sqrt{y^2 - v^2\,y^2})\,dy$$

$$yv\,dy + y^2dv = (vy + y\,\sqrt{1 - v^2})\,dy$$

$$yv\,dy + y^2dv = vy\,dy + (y\,\sqrt{1 - v^2})\,dy$$

$$y^2dv = y\,\sqrt{1 - v^2}\,dy$$

$$(\quad y^2dv = y\,\sqrt{1 - v^2}\,dy \quad) * \frac{1}{y^2\,\sqrt{1 - v^2}}$$

$$\frac{dv}{\sqrt{1 - v^2}} = \frac{dy}{y}$$

$$\int \frac{dv}{\sqrt{1 - v^2}} = \int \frac{dy}{y}$$

$$sin^{-1}(v) = ln\,y + c.$$

But $v = \dfrac{x}{y}$, so the solution is

$$sin^{-1}(\frac{x}{y}) = ln\,y + c.$$

Example 3. Check whether the following differential equation has homogeneous coefficients of the same degree or not; then find its solution

$$xy\,dx + (x^2 + y^2)\,dy = 0. \tag{2.16}$$

Solution: The coefficients in the equation are homogeneous and of degree two.
Let

$$x = v\,y, \qquad dx = v\,dy + y\,dv. \tag{2.17}$$

Put (2.17) in (2.16), yields

$$(\quad vy^2(v\,dy + y\,dv) + (v^2y^2 + y^2)\,dy = 0, \quad) * \frac{1}{y^2}$$

$$v(v\,dy + y\,dv) + (v^2 + 1)\,dy = 0$$

$$v^2\,dy + vy\,dv + v^2\,dy + dy = 0.$$

Hence we need to solve

$$(\quad vy\,dv + (2v^2 + 1)\,dy = 0, \quad) * \frac{1}{y(2v^2 + 1)}$$

we separate the variables such that

$$\int \frac{v}{2v^2+1}\,dv + \int \frac{dy}{y} = 0$$

$$ln(2v^2+1) + 4\ ln\ y = ln\ c$$

$$y^4(2v^2+1) = c.$$

Thus the desired solution is given by

$$y^4(\frac{2x^2}{y^2}+1) = c;$$

that is

$$y^2(2x^2+y^2) = c.$$

Exercises:

Find the solution of the following

1. $(x-2y)\,dx + (2x+y)\,dy = 0.$

2. $y\,dx = (x + \sqrt{y^2-x^2})\,dy$

3. $(y - \sqrt{x^2+y^2})\,dx - x\,dy = 0; \quad y(\sqrt{3}) = 1.$

4. $(3x^2-2y^2)\,y' = 2xy; \quad y(0) = -1.$

2.3 Exact Equation

A first-order DE

$$M(x,y)dx + N(x,y)dy = 0 \tag{2.18}$$

is said to be exact differential equation if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

The solution is $F(x,y) = c$ which satisfies

$$\frac{\partial F}{\partial x} = M, \quad \frac{\partial F}{\partial y} = N$$

(since $F(x,y) = c \implies \dfrac{\partial F}{\partial x}dx + \dfrac{\partial F}{\partial y}dy = 0$ and comparing with (2.18), yields $\dfrac{\partial F}{\partial x} = M, \quad \dfrac{\partial F}{\partial y} = N$).

Remark: In order to find the solution of the exact DE which is $F(x,y) = c$.
Since $\dfrac{\partial F}{\partial x} = M$ and $\dfrac{\partial F}{\partial y} = N$ hence we use one of them to find F such as

$$\frac{\partial F}{\partial x} = M \implies \int \frac{\partial F}{\partial x} = \int M\, dx \implies F(x,y) = \int M(x,y)\, dx + c(y)$$

and then using the other one $\dfrac{\partial F}{\partial y} = N$ to find $c(y)$.

Example 1. Solve

$$3x(xy-2)dx + (x^3+2y)dy = 0.$$

Solution: Clear that

$$M(x,y) = 3x^2y - 6x$$

$$N(x,y) = x^3 + 2y$$

and $\dfrac{\partial M}{\partial y} = 3x^2, \quad \dfrac{\partial N}{\partial x} = 3x^2 \implies \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ then the DE is exact.

Let $F(x,y) = c$ be the general solution.

$$\text{Since} \quad \frac{\partial F}{\partial x} = M$$

$$\implies \frac{\partial F}{\partial x} = 3x^2y - 6x \implies F(x,y) = \int (3x^2y - 6x)dx = x^3y - 3x^2 + c(y)$$

$$\text{then} \quad F(x,y) = x^3y - 3x^2 + c(y), \quad \text{to find} \quad c(y)$$

$$\text{since} \quad \frac{\partial F}{\partial y} = N$$

$$\Longrightarrow\ x^3 + c^{'}(y) = x^3 + 2y \ \Longrightarrow\ c^{'}(y) = 2y \ \Longrightarrow\ c(y) = \int 2y\, dy = y^2 + c.$$

Hence the general solution is

$$x^3 y - 3x^2 + y^2 = c.$$

Example 2. Solve the differential equation

$$(y\ cosx + 2x\ e^y)dx + (sinx + x^2 e^y - 1)dy = 0.$$

Solution:

$$M(x,y) = y\ cosx + 2x\ e^y \ \Longrightarrow\ \frac{\partial M}{\partial y} = cosx + 2xe^y$$

$$N(x,y) = sinx + x^2 e^y - 1 \ \Longrightarrow\ \frac{\partial N}{\partial x} = cosx + 2xe^y$$

then $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ $\Longrightarrow$ the DE is exact.

Now, let the solution be $F(x,y) = c$,

$$\text{since} \quad \frac{\partial F}{\partial x} = M \ \Longrightarrow\ F(x,y) = \int M(x,y)dx = \int (y\ cosx + 2x\ e^y)dx = y\ sinx + x^2 e^y + c(y)$$

$$\text{then} \quad F(x,y) = y\ sinx + x^2 e^y + c(y), \quad \text{to find} \quad c(y)$$

$$\text{since} \quad \frac{\partial F}{\partial y} = N$$

$$\Longrightarrow\ sinx + x^2 e^y + c^{'}(y) = sinx + x^2 e^y - 1 \ \Longrightarrow\ c^{'}(y) = -1 \ \Longrightarrow\ c(y) = -y + c.$$

Hence the general solution is

$$y\ sinx + x^2 e^y - y = c.$$

Example 3. Solve the equation

$$(2x^3 - xy^2 - 2y + 3)dx - (x^2 y + 2x)dy = 0.$$

Solution:

$$M(x,y) = 2x^3 - xy^2 - 2y + 3 \ \Longrightarrow\ \frac{\partial M}{\partial y} = -2xy - 2$$

$$N(x,y) = -x^2 y - 2x \ \Longrightarrow\ \frac{\partial N}{\partial x} = -2xy - 2$$

since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ then the DE is exact.

Let the solution is $F(x,y)=c$,
such that

$$\frac{\partial F}{\partial x}=M=2x^3-xy^2-2y+3 \tag{2.19}$$

and

$$\frac{\partial F}{\partial y}=N=-x^2y-2x. \tag{2.20}$$

Because (2.20) is simpler than (2.19), let us use eq.(2.20),

$$\frac{\partial F}{\partial y}=N \implies F(x,y)=\int N(x,y)dy=\int(-x^2y-2x)dy$$

$$\text{then} \quad F(x,y)=-\frac{1}{2}x^2y^2-2xy+c(x), \quad \text{to find} \quad c(x)$$

$$\text{since} \quad \frac{\partial F}{\partial x}=M$$

$$\implies -xy^2-2y+c^{'}(x)=2x^3-xy^2-2y+3 \implies c^{'}(x)=2x^3+3$$

$$\implies c(x)=\int(2x^3+3)dx=\frac{1}{2}x^4+3x+c.$$

Hence the general solution is

$$-\frac{1}{2}x^2y^2-2xy+\frac{1}{2}x^4+3x=c.$$

Example 4. Solve the diff. eq.

$$(2x+y^2)+2xy\,\frac{dy}{dx}=0.$$

Solution: The given equation is equivalent to

$$(2x+y^2)dx+2xy\,dy=0$$

Clearly

$$M(x,y)=2x+y^2 \implies \frac{\partial M}{\partial y}=2y$$

$$N(x,y)=2xy \implies \frac{\partial N}{\partial x}=2y$$

then $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$ $\implies$ the DE is exact.

Now, let the solution be $F(x,y)=c$,

$$\text{since} \quad \frac{\partial F}{\partial x}=M \implies F(x,y)=\int M(x,y)dx=\int(2x+y^2)dx$$

$$\text{then} \quad F(x,y) = x^2 + y^2x + c(y), \quad \text{to find} \quad c(y)$$

$$\text{since} \quad \frac{\partial F}{\partial y} = N$$

$$2yx + c^{'}(y) = 2xy \implies c^{'}(y) = 0 \implies c(y) = c.$$

Hence the general solution is

$$x^2 + y^2x = c.$$

Exercises:

Find the G.S or P.S. for the following differential equations

1. $x(3xy - 4y^3 + 6)dx + (x^3 - 6x^2y^2 - 1)dy = 0$
2. $(2xy\ cosx^2 - 2xy + 1)dx + (sinx^2 - x^2)dy = 0$
3. $(cos2y - 3x^2y^2)dx + (cos2y - 2x\ sin2y - 2x^3y)dy = 0$
4. $3y(x^2 - 1)dx + (x^3 + 8y - 3x)dy = 0, \quad y(0) = 1.$

2.4 Non-Exact ODE's Transformed to Exact ODE's (Determination of Integrating Factors)

Suppose that the diff. eq.

$$M(x,y)dx + N(x,y)dy = 0 \tag{2.21}$$

is not an exact differential equation, i.e., $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$ and let $u(x,y)$ be the integrating factor which transform the above differential equation to an exact differential equation. By multiplying both sides of (2.21) by $u(x,y)$, we have

$$uM\ dx + uN\ dy = 0$$

which must be exact, i.e.,

$$\begin{aligned}
&\frac{\partial}{\partial y}(uM) = \frac{\partial}{\partial x}(uN) \\
\implies\ & u\frac{\partial M}{\partial y} + M\frac{\partial u}{\partial y} = u\frac{\partial N}{\partial x} + N\frac{\partial u}{\partial x} \\
\implies\ & u\frac{\partial M}{\partial y} - u\frac{\partial N}{\partial x} = N\frac{\partial u}{\partial x} - M\frac{\partial u}{\partial y} \\
\implies\ & u(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = N\frac{\partial u}{\partial x} - M\frac{\partial u}{\partial y}.
\end{aligned} \tag{2.22}$$

Now, suppose we have the following two cases:

1. If $u = u(x)$ (function of x only). Then $\dfrac{\partial u}{\partial y} = 0$ and $\dfrac{\partial u}{\partial x} = \dfrac{du}{dx}$, therefore (2.22) will be

$$\begin{aligned}
&u(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = N\frac{du}{dx} \\
\implies\ & \frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})dx = \frac{du}{u} = f(x)
\end{aligned}$$

   i.e., $\displaystyle\int \frac{du}{u} = \int f(x)dx \implies ln(u) = \int f(x)dx \implies u(x) = e^{\int f(x)dx}$.

2. If $u = u(y)$ (function of y only). Then $\dfrac{\partial u}{\partial x} = 0$ and $\dfrac{\partial u}{\partial y} = \dfrac{du}{dy}$, and (2.22) will be

$$\begin{aligned}
&u(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = -M\frac{du}{dy} \\
\implies\ & \frac{1}{M}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})dy = -\frac{du}{u} = g(y)
\end{aligned}$$

   i.e., $\displaystyle\int \frac{du}{u} = -\int g(y)dy \implies ln(u) = -\int g(y)dy \implies u(y) = e^{-\int g(y)dy}$.

Example 1. Solve the following diff. eq.

$$(4xy + 3y^2 - x)dx + x(x + 2y)dy = 0 \tag{2.23}$$

Solution:

$$M(x, y) = 4xy + 3y^2 - x \implies \frac{\partial M}{\partial y} = 4x + 6y$$

$$N(x, y) = x^2 + 2xy \implies \frac{\partial N}{\partial x} = 2x + 2y$$

clear that $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the diff. eq. is not exact.

Now,

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 4x + 6y - 2x - 2y = 2x + 4y$$

then

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{2x + 4y}{x^2 + 2xy} = \frac{2(x + 2y)}{x(x + 2y)} = \frac{2}{x} = f(x).$$

Hence the integrating factor is

$$u(x) = e^{\int f(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\,lnx} = e^{lnx^2} = x^2$$

and therefore eq.(2.23) becomes $(4x^3y + 3x^2y^2 - x^3)dx + (x^4 + 2x^3y)dy = 0$

$$M(x, y) = 4x^3y + 3x^2y^2 - x^3 \implies \frac{\partial M}{\partial y} = 4x^3 + 6x^2y$$

$$N(x, y) = x^4 + 2x^3y \implies \frac{\partial N}{\partial x} = 4x^3 + 6x^2y$$

and $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \implies$ the diff. eq. is exact.

Let $F(x, y) = c$

$$\text{since} \quad \frac{\partial F}{\partial x} = M \implies F = \int (4x^3y + 3x^2y^2 - x^3)dx$$

$$\implies F(x, y) = x^4y + x^3y^2 - \frac{x^4}{4} + c(y)$$

$$\text{also since} \quad \frac{\partial F}{\partial y} = N \implies x^4 + 2x^3y + c'(y) = x^4 + 2x^3y$$

$$\implies c'(y) = 0 \implies c(y) = c.$$

Then the general solution is given by: $x^4y + x^3y^2 - \frac{x^4}{4} = c.$

Example 2. Solve

$$(3xy + y^2) + (x^2 + xy)\frac{dy}{dx} = 0$$

Solution: The given equation is equivalent to

$$(3xy + y^2)dx + (x^2 + xy)dy = 0 \tag{2.24}$$

$$M = 3xy + y^2 \implies \frac{\partial M}{\partial y} = 3x + 2y$$

$$N = x^2 + xy \implies \frac{\partial N}{\partial x} = 2x + y$$

since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the diff. eq. is not exact.

Now,

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 3x + 2y - 2x - y = x + y$$

then

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{x + y}{x^2 + xy} = \frac{x + y}{x(x + y)} = \frac{1}{x} = f(x).$$

Hence the integrating factor is

$$u(x) = e^{\int f(x)dx} = e^{\int \frac{dx}{x}} = x$$

and therefore eq.(2.24) becomes $(3x^2y + xy^2)dx + (x^3 + x^2y)dy = 0$

$$M(x, y) = 3x^2y + xy^2 \implies \frac{\partial M}{\partial y} = 3x^2 + 2xy$$

$$N(x, y) = x^3 + x^2y \implies \frac{\partial N}{\partial x} = 3x^2 + 2xy$$

and $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \implies$ the diff. eq. is exact.

Let $F(x, y) = c$

$$\text{since} \quad \frac{\partial F}{\partial x} = M \implies F = \int (3x^2y + xy^2)dx$$

$$\implies F(x, y) = x^3y + \frac{1}{2}x^2y^2 + c(y)$$

$$\text{also since} \quad \frac{\partial F}{\partial y} = N \implies x^3 + x^2y + c'(y) = x^3 + x^2y \implies c(y) = c.$$

Then the general solution is given by: $x^3y + \frac{1}{2}x^2y^2 = c$.

Example 3. Solve the equation

$$y(x+y+1)dx + x(x+3y+2)dy = 0 \tag{2.25}$$

Solution:

$$M(x,y) = xy + y^2 + y \implies \frac{\partial M}{\partial y} = x + 2y + 1$$

$$N(x,y) = x^2 + 3xy + 2x \implies \frac{\partial N}{\partial x} = 2x + 3y + 2$$

clear that $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, then the diff. eq. is not exact.

Now,

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = x + 2y + 1 - 2x - 3y - 2 = -x - y - 1$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{-x-y-1}{x(x+3y+2)} \neq f(x)$$

$$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{-x-y-1}{y(x+y+1)} = \frac{-(x+y+1)}{y(x+y+1)} = -\frac{1}{y} = g(y)$$

Hence the integrating factor is

$$u(y) = e^{-\int g(y)dy} = e^{-\int -\frac{1}{y}dy} = e^{\int \frac{dy}{y}} = e^{lny} = y$$

and therefore eq.(2.25) becomes $(xy^2 + y^3 + y^2)dx + (x^2y + 3xy^2 + 2xy)dy = 0$

$$M(x,y) = xy^2 + y^3 + y^2 \implies \frac{\partial M}{\partial y} = 2xy + 3y^2 + 2y$$

$$N(x,y) = x^2y + 3xy^2 + 2xy \implies \frac{\partial N}{\partial x} = 2xy + 3y^2 + 2y$$

and $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \implies$ the diff. eq. is exact.

Let $F(x,y) = c$ be the solution

$$\text{since} \quad \frac{\partial F}{\partial x} = M \implies F(x,y) = \int (xy^2 + y^3 + y^2)dx$$

$$\implies F(x,y) = \frac{x^2y^2}{2} + xy^3 + xy^2 + c(y)$$

$$\text{to find } c(y), \text{ since} \quad \frac{\partial F}{\partial y} = N \implies x^2y + 3xy^2 + 2xy + c'(y) = x^2y + 3xy^2 + 2xy$$

$$\implies c(y) = c$$

Hence the general solution is

$$\frac{x^2y^2}{2} + xy^3 + xy^2 = c.$$

Example 4. Solve the equation

$$y(x+y)dx + (x+2y-1)dy = 0 \tag{2.26}$$

Solution:

$$M = yx + y^2 \implies \frac{\partial M}{\partial y} = x + 2y$$

$$N = x + 2y - 1 \implies \frac{\partial N}{\partial x} = 1$$

since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the diff. eq. is not exact.

Now,

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = x + 2y - 1$$

then

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{x+2y-1}{x+2y-1} = 1 = f(x).$$

Hence the integrating factor is

$$u(x) = e^{\int f(x)dx} = e^{\int dx} = e^x$$

and therefore eq.(2.26) becomes $(xye^x + y^2e^x)dx + (xe^x + 2ye^x - e^x)dy = 0$

$$M(x,y) = xye^x + y^2e^x \implies \frac{\partial M}{\partial y} = xe^x + 2ye^x$$

$$N(x,y) = xe^x + 2ye^x - e^x \implies \frac{\partial N}{\partial x} = xe^x + e^x + 2ye^x - e^x = xe^x + 2ye^x$$

and $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \implies$ the diff. eq. is exact.

Let $F(x,y) = c$

$$\text{since} \quad \frac{\partial F}{\partial x} = M \implies F = \int (xye^x + y^2e^x)dx$$

$$\implies F(x,y) = y(xe^x - e^x) + y^2e^x + c(y)$$

$$\text{also since} \quad \frac{\partial F}{\partial y} = N \implies xe^x - e^x + 2ye^x + c'(y) = xe^x + 2ye^x - e^x \implies c(y) = c.$$

Then the general solution is given by: $y(xe^x - e^x) + y^2e^x = c$.

Exercises:

Solve the following differential equations

1. $(xy+1)dx + (x^2+4xy-2x)dy = 0, \quad y(1) = 2$
2. $(2x^2y - xy^2 + y)dx + (x-y)dy = 0, \quad y(0) = -4$
3. $(x^2+y^2+1)dx + (x^2-2xy)dy = 0$
4. $(2y^2+3xy-2y+6x)dx + (x^2+2xy-x)dy = 0.$

2.5 Linear First-Order Equation

The first-order DE

$$F(x, y, \frac{dy}{dx}) = 0 \tag{2.27}$$

is said to be linear equation if it has the form

$$\frac{dy}{dx} + p(x)y = Q(x). \tag{2.28}$$

Remark: One can see easily that the linear equation is an equation which is non-exact and can be transformed to an exact equation using the integrating factor $u(x) = e^{\int p(x)dx}$ and also can easily show that the general solution is given by

$$yu = \int u(x)Q(x)dx$$

or

$$y = u^{-1} \int u(x)Q(x)dx.$$

Example 1. Solve the initial value problem

$$\frac{dy}{dx} - y = \frac{11}{8}\, e^{\frac{-x}{3}}, \quad y(0) = -1. \tag{2.29}$$

solution: Here we have $p(x) = -1$ and $Q(x) = \dfrac{11}{8}\, e^{\frac{-x}{3}}$,
so the integrating factor is

$$u(x) = e^{\int p(x)\, dx} = e^{\int (-1)\, dx} = e^{-x}.$$

The G.S. is

$$yu = \int u(x)Q(x)dx$$

$$y\, e^{-x} = \int \frac{11}{8}\, e^{\frac{-4x}{3}}\, dx = \frac{-33}{32}\, e^{\frac{-4x}{3}} + c,$$

or

$$y(x) = ce^{x} - \frac{33}{32}\, e^{\frac{-x}{3}}.$$

Substitution of $x = 0$ and $y = -1$ now gives $c = \dfrac{1}{32}$, so the desired particular solution is

$$y(x) = \frac{1}{32}\, e^{x} - \frac{33}{32}\, e^{\frac{-x}{3}} = \frac{1}{32}\left(e^{x} - 33\, e^{\frac{-x}{3}}\right).$$

Example 2. Find the general solution of

$$(x^2+1)\,\frac{dy}{dx}+3xy=6x. \tag{2.30}$$

Solution: After division of both sides of the equation by x^2+1, we recognize the result

$$\frac{dy}{dx}+\frac{3x}{x^2+1}\,y=\frac{6x}{x^2+1}$$

as a first-order linear equation with $p(x)=\dfrac{3x}{x^2+1}$ and $Q(x)=\dfrac{6x}{x^2+1}$.

$$\int p(x)\,dx=\int\frac{3x}{x^2+1}\,dx=\frac{3}{2}\,ln(x^2+1)$$

$$u(x)=e^{\int p(x)\,dx}=e^{\frac{3}{2}\,ln(x^2+1)}=(x^2+1)^{\frac{3}{2}}.$$

The G.S. is

$$y\,u(x)=\int u(x)Q(x)dx$$

$$y\,(x^2+1)^{\frac{3}{2}}=\int 6x\,(x^2+1)^{\frac{1}{2}}\,dx=2(x^2+1)^{\frac{3}{2}}+c.$$

Example 3. Find the general solution of

$$y^{'}=1+x+y+x\,y.$$

Solution:

$$\frac{dy}{dx}=1+x+y+x\,y$$

$$\frac{dy}{dx}-(1+x)\,y=1+x, \tag{2.31}$$

which is a linear first-order equation with

$$p(x)=-(1+x),\quad Q(x)=1+x.$$

$$\int p(x)\,dx=\int-(1+x)\,dx=-(x+\frac{x^2}{2})$$

$$u=e^{\int p(x)\,dx}=e^{-(x+\frac{x^2}{2})}.$$

The G.S. is

$$y\,u=\int u(x)Q(x)dx$$

$$y\,e^{-(x+\frac{x^2}{2})}=\int(1+x)\,e^{-(x+\frac{x^2}{2})}\,dx$$

$$y\,e^{-(x+\frac{x^2}{2})}=-e^{-(x+\frac{x^2}{2})}+c.$$

Note: Sometimes the linear equation may has the form

$$\frac{dx}{dy} + p(y)x = Q(y), \tag{2.32}$$

instead of eq. (2.28). In this case, the solution of (2.32) is

$$xu = \int u(y)\ Q(y)dy, \text{ where } \quad u(y) = e^{\int p(y)dy}.$$

Example 4. Solve the following equation

$$\frac{dx}{dy} + (\frac{3}{y} - 1)x = -\frac{2}{y}.$$

Solution:

$$p(y) = \frac{3}{y} - 1, \quad Q(y) = -\frac{2}{y}.$$

$$\int p(y)\ dy = \int (\frac{3}{y} - 1)dy = 3ln(y) - y = lny^3 - y$$

$$u(y) = e^{\int p(y)\ dy} = e^{lny^3 - y} = e^{lny^3}e^{-y} = y^3e^{-y}.$$

The G.S. is

$$x\ u(y) = \int u(y)Q(y)dy$$

$$xy^3e^{-y} = \int y^3e^{-y}.(-\frac{2}{y})dy$$

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Applications

Mixture Problems: As an application of linear first-order equations, we consider a tank containing a solution (a mixture of solute and solvent) such as salt dissolved in water. There is both inflow and outflow, and we want to compute the amount $x(t)$ of solute in the tank at time t, given the amount $x(0) = x_0$ at time $t = 0$. Suppose that solution with a concentration of c_i grams of solute per liter of solution flows into the tank at the constant rate of r_i liters per second, and that the solution in the tank (kept thoroughly mixed by stirring) flows out at the constant rate of r_o liters per second.
To set up a differential equation for $x(t)$, we estimate the change Δx in x during the brief time interval $[t, t + \Delta t]$. The amount of solute that flows into the tank during Δt seconds is $r_i c_i \Delta t$ grams.
The amount of solute that flows out of the tank during the same time interval depends on the concentration $c_o(t)$ of solute in the solution at time t. But as noted in Fig. 2.1,

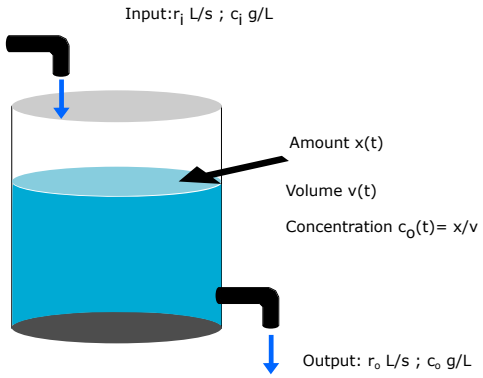

Figure 2.1: The single-tank mixture problem

$c_o(t) = \dfrac{x(t)}{v(t)}$, where $v(t)$ denotes the volume (not constant unless $r_i = r_o$) of solution in the tank at time t.

Then

$$\Delta x = \text{grams input} - \text{grams output} \approx r_i c_i \Delta t - r_o c_o \Delta t.$$

We now divide by Δt:

$$\frac{\Delta x}{\Delta t} \approx r_i c_i - r_o c_o.$$

Finally, we take the limit as $\Delta t \to 0$; if all the functions involved are continuous and $x(t)$ is differentiable, then the error in this approximation also approaches zero, and we obtain the differential equation

$$\frac{dx}{dt} = r_i c_i - r_o c_o, \tag{2.33}$$

in which r_i, c_i and r_o are constants, but c_o denotes the variable concentration

$$c_o(t) = \frac{x(t)}{v(t)} \tag{2.34}$$

of solute in the tank at time t. Thus the amount $x(t)$ of solute in the tank satisfies the differential equation

$$\frac{dx}{dt} = r_i c_i - \frac{r_o}{v} x, \tag{2.35}$$

which is a linear first-order differential equation for the amount $x(t)$ of solute in the tank at time t.

Example 4. Assume that Lake Erie has a volume of 480 km^3 and that its rate of inflow (from Lake Huron) and outflow (to Lake Ontario) are both 350 km^3 per year. Suppose that at the time $t = 0$ (years), the pollutant concentration of Lake Erie (caused by past industrial pollution that has now been ordered to cease) is five times that of Lake Huron. If the outflow, henceforth is perfectly mixed lake water, how long will it take to reduce the pollution concentration in Lake Erie to twice that of Lake Huron?

Solution: Here we have

$$v = 480\ (km^3),$$

$$r_i = r_o = r = 350\ (\frac{km^3}{yr}),$$

$$c_i = c \text{ (the pollutant concentration of Lake Huron)}$$

$$\text{and} \quad x_0 = x(0) = 5\ cv, \quad \text{where } 5c = \frac{x_0}{v}$$

and the question is this: When is $x(t) = 2cv$? with this notation, eq.(2.35) is the equation

$$\frac{dx}{dt} = rc - \frac{r}{v}\ x,$$

which we rewrite in the linear first-order form

$$\frac{dx}{dt} + \frac{r}{v}x = rc \tag{2.36}$$

with constant coefficients $p = \frac{r}{v},\ q = rc,$

and integrating factor $u = e^{\int p\ dt} = e^{pt} = e^{(\frac{r}{v})\ t}$.
Then the solution of equation (2.36) is

$$x(t)u = \int u(t)q\ dt$$

$$x(t)e^{(\frac{r}{v})\ t} = \int e^{(\frac{r}{v})\ t}\ rc\ dt$$

$$x(t)e^{(\frac{r}{v})\ t} = e^{(\frac{r}{v})\ t}\ cv + \bar{c}$$

$$x(t) = e^{-(\frac{r}{v})\ t}\Big(e^{(\frac{r}{v})\ t}\ cv + \bar{c}\Big)$$

$$x(t) = cv + e^{-(\frac{r}{v})\ t}\ \bar{c}. \tag{2.37}$$

Substitute the initial condition $x(0) = 5\ cv$, (i.e. at $t = 0,\ x = 5\ cv$) gives

$$5\ cv = cv + \bar{c} \implies \bar{c} = 4\ cv.$$

Substitute the value of $\bar{c}$ in (2.37), yields

$$x(t) = cv + 4\ cv\ e^{-(\frac{r}{v})\ t}. \tag{2.38}$$

To find when $x(t) = 2\ cv$, we therefore need only to solve eq. (2.38) for t after substituting $x(t) = 2\ cv$ as follows:

$$cv + 4\ cv\ e^{-(\frac{r}{v})\ t} = 2\ cv$$

$$\implies t = \frac{v}{r}\ ln4 = \frac{480}{350}\ ln4 \approx 1.901\text{years}.$$

Example 5. A 120-litres (ltr) tank initially contains 90 gram (grm) of salt dissolved in 90 ltr of water. Brine containing 2 grm/ltr of salt flows into the tank at the rate of 4 ltr/min, and the well-stirred mixture flows out of the tank at the rate of 3 ltr/min. How much salt does the tank contain when it is full?

Solution: The interesting feature of this example is that due to the differing rates of inflow and outflow, the volume of brine in the tank increases steadily with $v(t) = 90 + t$ gallons. The change Δx in the amount x of salt in the tank from time t to time $t + \Delta t$ (minutes) is given by

$$\Delta x \approx (4)(2)\Delta t - 3(\frac{x}{90+t})\Delta t,$$

so our differential equation is

$$\frac{dx}{dt} + \frac{3}{90+t}\, x = 8,$$

with $P(t) = \dfrac{3}{90+t}$ and $Q(t) = 8$.

$$\int p(t)dt = \int \frac{3}{90+t}\, dt = 3\, ln(90+t)$$

An integrating factor is

$$u(t) = e^{p(t)dt} = e^{3\, ln(90+t)} = (90+t)^3.$$

The G.S. is

$$x(t)u = \int u(t)Q(t)dt$$

$$x(t)\ (90+t)^3 = 8\int (90+t)^3 dt$$

$$x(t)\ (90+t)^3 = 2(90+t)^4 + c$$

$$x(t) = (90+t)^{-3}\Big(2(90+t)^4 + c\Big) \implies x(t) = 2(90+t) + \frac{c}{(90+t)^3}$$

Substitution of $x(0) = 90$ gives $c = -(90)^4$, so the amount of salt in the tank at time t is

$$x(t) = 2(90+t) - \frac{90^4}{(90+t)^3}.$$

The tank is full after 30 min, and when $t = 30$, we have

$$x(30) = 2(90+30) - \frac{90^4}{120^3} \approx 202\ (lb)$$

of salt in the tank.

Exercises:

Find the G.S of the following problems. If an initial condition is given, then find the corresponding P.S.

1. $xy' - y = x, \quad y(1) = 7$
2. $xy' = 2y + x^3 cosx$
3. $(x^2 + 4)y' + 3xy = x, \quad y(0) = 1$
4. $2(y - 4x^2)dx + xdy = 0.$

2.6 Bernoulli's Equation

A first-order DE of the form

$$\frac{dy}{dx} + p(x)y = Q(x)y^n. \tag{2.39}$$

is called a Bernoulli equation.

Remark: If $n = 1$, then eq.(2.39) transformed to a separable differential equation, and if $n \neq 1$ then Bernoulli equation (2.39) transformed to a linear equation using the transformation

$$z = y^{1-n} \implies dz = (1 - n)y^{-n}\, dy.$$

If we multiply (2.39) by $(1 - n)y^{-n}$ such that

$$(1 - n)y^{-n}\frac{dy}{dx} + (1 - n)p(x)y^{1-n} = (1 - n)Q(x)$$

and substitute $z = y^{1-n}$, $dz = (1 - n)y^{-n}\, dy$ then, this yields the linear equation

$$\frac{dz}{dx} + (1 - n)p(x)z = (1 - n)Q(x). \tag{2.40}$$

Therefore, the Bernoulli equation (2.39) is transformed to the linear equation (2.40).

After solving the linear equation, return to the original variables y and x instead of z and x.

Example 1. Solve the equation

$$\frac{dy}{dx} - \frac{3}{2x}y = \frac{2x}{y}.$$

Solution:
The DE is Bernoulli equation with $p(x) = -\frac{3}{2x}$, $Q(x) = 2x$ and $n = -1$.
Let

$$z = y^2 \implies dz = 2y\, dy$$

substituting in Bernoulli equation, yields the liner equation

$$\frac{dz}{dx} - 2\left(\frac{3}{2x}\right)z = 2(2x) \implies \frac{dz}{dx} - \frac{3}{x}z = 4x$$

with $p(x) = -\frac{3}{x}$, $Q(x) = 4x$ and integrating factor

$$u = e^{\int -\frac{3}{x}dx} = e^{-3\,ln(x)} = x^{-3}.$$

Hence the G.S. is

$$zu = \int u(x)Q(x)dx$$

$$zx^{-3} = 4\int x^{-2}dx \implies zx^{-3} = -\frac{4}{x} + c$$

But $z = y^2 \implies$ the G.S. is

$$y^2x^{-3} = -\frac{4}{x} + c \quad \text{or} \quad y^2 = -4x^2 + c.$$

Example 2. Solve

$$y(6y^2 - x - 1) + 2x\frac{dy}{dx} = 0.$$

Solution:

$$\{ \qquad 2x\frac{dy}{dx} + 6y^3 - (x+1)y = 0 \qquad \} * \frac{1}{2x}$$

$$\implies \frac{dy}{dx} - \frac{x+1}{2x}y = -\frac{3}{x}y^3$$

which is Bernoulli equation.
Let

$$z = y^{-2} \implies dz = -2y^{-3}\, dy$$

substituting in Bernoulli equation, yields the liner equation

$$\frac{dz}{dx} - (-2)\frac{x+1}{2x}\, z = -(-2)\frac{3}{x} \implies \frac{dz}{dx} + \frac{x+1}{x}\, z = \frac{6}{x}$$

with $p(x) = \dfrac{x+1}{x}$, $Q(x) = \dfrac{6}{x}$ and integrating factor

$$u = e^{\int \frac{x+1}{x} dx} = e^{\int (1+\frac{1}{x}) dx} = e^{x+ln(x)} = xe^x.$$

Hence the G.S. is

$$zu = \int u(x)Q(x)dx$$

$$z\ xe^x = \int xe^x.\frac{6}{x}dx \implies zxe^x = 6\int e^x dx$$

$$\implies zxe^x = 6e^x + c.$$

But $z = y^{-2} \implies$ the G.S. is

$$y^{-2}xe^x = 6e^x + c.$$

Exercises:

Solve the following DEs

1. $2x^3y' = y(y^2 + 3x^2)$
2. $\dfrac{dy}{dx} + \dfrac{y}{x} = xy^3$
3. $(x^2 + 6y^2) - 4xy\ \dfrac{dy}{dx} = 0, \quad y(1) = 1.$

2.7 Integrating Factors Found by Inspection

We shall concern with DE.'s that are simple enough to enable us to find integrating factors by inspection. The ability depends on recognition of certain common exact differentials that occur frequently and experience. The following are four exact differentials:

$$(1)\quad d(xy) = xdy + ydx \qquad (2)\quad d(\frac{x}{y}) = \frac{ydx - xdy}{y^2}$$

$$(3)\quad d(\frac{y}{x}) = \frac{xdy - ydx}{x^2} \qquad (4)\quad d(tan^{-1}\,\frac{y}{x}) = \frac{xdy - ydx}{x^2 + y^2}.$$

Example 1. Find the G.S. of the following differential equation

$$ydx + (x + x^3y^2)dy = 0.$$

Solution:

$$ydx + xdy + x^3y^2dy = 0 \implies d(xy) = -x^3y^2dy$$

$$\implies d(xy) = -\frac{(xy)^3}{y}dy \implies \frac{d(xy)}{(xy)^3} = -\frac{dy}{y}$$

$$\implies \int \frac{d(xy)}{(xy)^3} = -\int \frac{dy}{y} \implies -\frac{1}{2(xy)^2} = -ln(y) + c$$

or

$$ln(y) - \frac{1}{2x^2y^2} = c.$$

Example 2. Solve

$$y(x^3 - y)dx - x(x^3 + y)dy = 0.$$

Solution:

$$yx^3dx - y^2dx - x^4dy - xydy = 0$$

$$\implies x^3(ydx - xdy) = y(ydx + xdy)$$

$$x^3y^2\frac{ydx - xdy}{y^2} = y(ydx + xdy)$$

$$\{ \qquad \implies x^3y^2d(\frac{x}{y}) = y\;d(xy) \qquad \} * \frac{1}{x^2y^3}$$

$$\implies (\frac{x}{y})d(\frac{x}{y}) = \frac{d(xy)}{(xy)^2}$$

$$\int \frac{x}{y}\;d(\frac{x}{y}) = \int \frac{d(xy)}{(xy)^2} \implies \frac{1}{2}(\frac{x}{y})^2 - \frac{1}{xy} + c.$$

Exercises:

Solve the following DEs

1. $y(2xy+1)dx - xdy = 0$
2. $y(y^3 - x)dx + x(y^3 + x)dy = 0$
3. $x(x^2 - y^2 - x)dx - y(x^2 - y^2)dy = 0, \quad y = 2,\ y = 0.$

Chapter 3

Linear Differential Equations: The General Form of Linear Equation

The general linear differential equation of order n is an equation that can be written as:

$$b_0(x)\frac{d^n y}{dx^n} + b_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + b_{n-1}(x)\frac{dy}{dx} + b_n(x)y = R(x) \qquad (1)$$

The functions $R(x)$ and $b_i(x), i = 0, \cdots, n$ are to be independent of variable y. If $b_0, b_1, \cdots, b_n$ are constants numbers, then eq. (1) is said to be a linear equation of order n with constant coefficients.

If $R(x)$ is identically zero, then eq. (1) is said to be linear and homogeneous; if $R(x)$ is not identically zero, then eq. (1) is said to be linear and non-homogeneous.

In this chapter we shall obtain some fundamental and important properties of linear equations.

Theorem 1. If y_1 and y_2 are solutions of the homogeneous equation:

$$b_0(x)\frac{d^n y}{dx^n} + b_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + b_{n-1}(x)\frac{dy}{dx} + b_n(x)y = 0 \qquad (2)$$

and if c_1 and c_2 are constants, then $y = c_1\, y_1 + c_2\, y_2$ is also a solution of eq. (2).

Proof: Since y_1 and y_2 are solutions of eq. (2), then

$$b_0(x)\frac{d^n y_1}{dx^n} + b_1(x)\frac{d^{n-1}y_1}{dx^{n-1}} + \cdots + b_{n-1}(x)\frac{dy_1}{dx} + b_n(x)y_1 = 0$$

and

$$b_0(x)\frac{d^n y_2}{dx^n} + b_1(x)\frac{d^{n-1}y_2}{dx^{n-1}} + \cdots + b_{n-1}(x)\frac{dy_2}{dx} + b_n(x)y_2 = 0$$

Now

$b_0(x)\frac{d^n}{dx^n}(c_1\, y_1 + c_2\, y_2) + b_1(x)\frac{d^{n-1}}{dx^{n-1}}(c_1\, y_1 + c_2\, y_2) + \cdots + b_{n-1}(x)\frac{d}{dx}(c_1\, y_1 + c_2\, y_2) + b_n(x)(c_1\, y_1 + c_2\, y_2) \stackrel{?}{=} 0$

$$= b_0(x)\, c_1\, \frac{d^n y_1}{dx^n} + b_0(x)\, c_2\, \frac{d^n y_2}{dx^n} + b_1(x)\, c_1\, \frac{d^{n-1} y_1}{dx^{n-1}} + b_1(x)\, c_2\, \frac{d^{n-1} y_2}{dx^{n-1}} + \cdots$$
$$+ b_{n-1}(x)\, c_1\, \frac{dy_1}{dx} + b_{n-1}(x)\, c_2\, \frac{dy_2}{dx} + b_n(x)\, c_1\, y_1 + b_n(x)\, c_2\, y_2$$
$$= c_1 \left[b_0(x)\, \frac{d^n y_1}{dx^n} + b_1(x)\, \frac{d^{n-1} y_1}{dx^{n-1}} + \cdots + b_{n-1}(x)\, \frac{dy_1}{dx} + b_n(x)\, y_1 \right]$$
$$+ c_2 \left[b_0(x)\, \frac{d^n y_2}{dx^n} + b_1(x)\, \frac{d^{n-1} y_2}{dx^{n-1}} + \cdots + b_{n-1}(x)\, \frac{dy_2}{dx} + b_n(x)\, y_2 \right]$$
$$= c_1 . 0 + c_2 . 0 = 0.$$

$\therefore y = c_1\, y_1 + c_2\, y_2$ is also a solution of eq. (2).

Remark: In a similar manner, it can be seen that if $y_i, i = 1, \cdots, k$ are solutions of eq. (2), and if $c_i, i = 1, \cdots, k$ are constants. Then

$$y = c_1\, y_1 + c_2\, y_2 + \cdots + c_k\, y_k \qquad (3)$$

is also a solution of eq. (2).

The expression in (3) is called a linear combination of the functions $y_1, y_2, \cdots, y_k$.

Usually, we say that any linear combination of solutions for a linear homogeneous differential equation is also a solution.

Remark: (Solution of Homogeneous Equation)

If $y_1, y_2, \cdots, y_n$ are n linearly independent solutions of the n-th order homogeneous linear differential equation with constant coefficients then there exist constants $c_1, c_2, \cdots, c_n$ such that

$$y = c_1\, y_1 + c_2\, y_2 + \cdots + c_n\, y_n$$

is the general solution of the differential equation.

Example The homogeneous DE

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0 \qquad (4)$$

has solutions $y = e^{-2x}$ and $y = e^{-3x}$. Check that $y = c_1\, e^{-2x} + c_2\, e^{-3x}$ is also a solution where c_1 and c_2 are constants.

Solution: $$y = c_1\, e^{-2x} + c_2\, e^{-3x} \qquad (5)$$

$$\Rightarrow y' = -2c_1\, e^{-2x} - 3c_2\, e^{-3x}$$

$$\Rightarrow y'' = 4c_1\, e^{-2x} + 9c_2\, e^{-3x}$$

$$\therefore y'' + 5y' + 6y = 4c_1 e^{-2x} + 9c_2 e^{-3x} - 10c_1 e^{-2x} - 15c_2 e^{-3x} + 6c_1 e^{-2x} + 6c_2 e^{-3x} = 0$$

$\therefore$ (5) is a solution of (4).

Exercises:

1. The DE $2x^2 \frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - y = 0$ has solutions $y = x^{\frac{1}{2}}$ and $y = x^{-1}$. Check that

$$y = c_1\, x^{\frac{1}{2}} + c_2\, x^{-1}$$

   is also a solution where c_1 and c_2 are constants.

2. Consider $y'' + y = 0$ and $y_1 = \sin x$, $y_2 = \cos x$, are two solutions to the given DE. Show that $c_1\, y_1 + c_2\, y_2$ is also a solution to the DE $y'' + y = 0$.

Linear Combinations:

Given some functions $f_1(x), f_2(x), \cdots, f_n(x)$ and $c_1, c_2, \cdots, c_n$ are constants, then the linear combination of $f_1(x), f_2(x), \cdots, f_n(x)$ is given by:

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x).$$

Linear Dependent:

The n functions $f_1(x), f_2(x), \cdots, f_n(x)$ are said to be linearly dependent if there exist constants $c_1, c_2, \cdots, c_n$ such that $c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$ implies that some or all of $c_i^{'s}$, $i = 1,2, \cdots, n$ not equal to zero.

Example:

1. The two functions $f_1(x) = x,\ f_2(x) = 2x$ are linearly dependent since there exist $c_1 = 2$, $c_2 = -1$ s.t.
$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0\,, \quad c_1 \neq 0, \quad c_2 \neq 0.$$

2. Let $f_1(x) = sinx,\ f_2(x) = 3sinx,\ f_3(x) = -sinx$, then there exist constants c_1, c_1 and c_3 not equal zero and $c_1 = 1,\ c_2 = 1, c_3 = 4$ and $c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = sinx + 3sinx - 4sinx = 0$ and therefore, f_1, f_2 and f_3 are linearly dependent.

Linear Independent:

The n-functions $f_1(x), f_2(x), \cdots, f_n(x)$ are called linearly independent if $c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$ implies that $c_1 = c_2 = \cdots = c_n = 0$.

Examples:

1. The two functions $f_1(x) = e^x,\ f_2(x) = e^{2x}$ are linearly independent. Since if we let $c_1 f_1 + c_2 f_2 = 0 \quad \rightarrow \quad c_1 e^x + c_2 e^{2x} = 0$ and since $e^x \neq 0,\ e^{2x} \neq 0 \quad \rightarrow \quad c_1 = c_1 = 0$.
2. Let $f_1(x) = sinx$, $f_2(x) = cosx$, then if we let
$c_1 f_1 + c_2 f_2 = 0 \quad \rightarrow \quad c_1 sinx + c_2 cosx = 0$
Since: $sin\theta \neq 0$ and $cos\theta \neq 0$ at the same angle θ, therefore $c_1 = c_1 = 0$.

For sets of n functions $f_1(x), f_2(x), \cdots, f_n(x)$, the above tests of independency become so difficult, and hence we use another toll for testing the independency of $f_1(x), f_2(x), \cdots, f_n(x)$ which is based on the following definition.

The Wronskian: The Wronskian of the n functions $f_1(x), f_2(x), \cdots, f_n(x)$ is the determinant

$$w(f_1, f_2, \cdots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & & & \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

Remark: If $w(f_1, f_2, \cdots, f_n) \neq 0$, then it follows that $c_1 = c_2 = \cdots = c_n = 0$, i.e., the set of functions $f_1(x), f_2(x), \cdots, f_n(x)$ are linearly independent.

Examples: Test the following for dependent and independent:

1. $f_1(x) = e^x,\ f_2(x) = e^{2x}\ ,\ f_3(x) = e^{3x}$.
2. $f_1(x) = 1,\ f_2(x) = sinx\ ,\ f_3(x) = cosx$.
3. $f_1(x) = 1,\ f_2(x) = sin^2x\ ,\ f_3(x) = cos^2x$.
4. $f_1(x) = x,\ f_2(x) = 2x\ .$

Solution:

1. $w(x) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^x(18e^{5x} - 12e^{5x}) + e^{2x}(3e^{4x} - 9e^{4x}) +$
   $e^{3x}(4e^{3x} - 2e^{3x}) = 6e^{6x} - 6e^{6x} + 2e^{6x} = 2e^{6x}$

   $\because\ e^{6x} \neq 0 \quad \rightarrow \quad w(x) \neq 0$

   $\therefore\ f_1, f_2, f_3$ are linearly independent.

2. $w(x) = \begin{vmatrix} 1 & sinx & cosx \\ 0 & cosx & -sinx \\ 0 & -sinx & -cosx \end{vmatrix} = -cos^2x - sin^2x + sinx(0 - 0) + cosx(0 + 0)$
   $= -(sin^2x + cos^2x) = -1 \neq 0$

   $\therefore\ f_1, f_2, f_3$ are linearly independent.
3. H.W.
4. H.W.

Remark: If $y_1, y_2, \cdots, y_n$ are solutions of the homogeneous DE. (2) then a necessary and sufficient condition that $y_1, y_2, \cdots, y_n$ be linearly independent is the non-vanishing of the wronskian of $y_1, y_2, \cdots, y_n$.

Example: Show that the functions $y_1 = cosx,\ y_2 = sinx,\ y_3 = \sin(x + 2)$ are linearly dependent.

Solution: …….
…….
…….

…….

Example: Show that the functions $1, x, x^2, x^3, \cdots, x^{n-1}$ are linearly independent.

Solution: $w = |\quad| = 0!\,1!\,2!\,3!\,\cdots(n-1)!\ \neq 0.$
$\therefore$ The given functions are linearly independent.

Example: The functions $y_1 = e^{-2x},\ y_2 = e^{-3x}$ are solutions of DE $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0.$
Find the Wronskian of y_1 and y_2 and show that they are linearly independent.

Solution: The wronskian of these two functions is

$$w = \begin{vmatrix} e^{-2x} & e^{-3x} \\ -2e^{-2x} & -3e^{-3x} \end{vmatrix} = -e^{-5x} \neq 0.$$

Since $w \neq 0 \ \ \therefore \ \ y_1$ and y_2 are linearly independent.

Exercises:

1. Show that the functions e^x, e^{2x}, e^{3x} are linearly independent.
2. Show that the functions $x, e^x, xe^x, (2-3x)e^x$ are linearly dependent and determine the constants c_1, c_2, c_3, c_4 which are not all zero such that
$$c_1x + c_2e^x + c_3xe^x + c_4(2-3x)e^x = 0.$$
3. Show that the functions $1, sin^2x, cos^2x$ are linearly dependent.

The Homogeneous Linear Equation with Constant Coefficients:

The general form of a homogeneous linear equation with constant coefficients is given by

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = 0, \qquad (1)$$

$a_n \neq 0$ and $a's$ are constants.

In order to find the general solution to the DE (1), let

$$D = \frac{d}{dx} \ , \qquad D^2 = \frac{d^2}{dx^2} \ , \cdots , \qquad D^n = \frac{d^n}{dx^n}$$

and the equation will transform to

$$a_n D^n y + a_{n-1} D^{n-1} y + \cdots + a_1 D y + a_0 y = 0$$

$$\Rightarrow (a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0) y = 0$$

i.e. $f(D)y = 0, \quad f(D) = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0.$

And if m is any root of the n-th degree polynomial.

$f(m) = 0$ which is called the characteristic equation associated with the n-th order linear differential equation with constant coefficients, and we have the following cases:

1. $f(m) = 0$ has n distinct (different) real roots.
2. $f(m) = 0$ has real repeated (equaled) roots.
3. $f(m) = 0$ has imaginary (complex) roots.

1. **Real Different Roots**:

If $f(m) = 0$ has n distinct real roots $m_1, m_2, \cdots, m_n$. Then we let $y_1(x) = e^{m_1 x}$, $y_2(x) = e^{m_2 x}, \cdots, y_n(x) = e^{m_n x}$ to be the linearly independent solution therefore the general solution is given by:

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}.$$

Example Solve $y'' + 2y' - 3y = 0$.

Solution:

$$y'' + 2y' - 3y = 0 \quad \Rightarrow \quad f(D) = D^2 + 2D - 3$$

The auxiliary equation $f(m) = 0 \quad \Rightarrow \quad m^2 + 2m - 3 = 0 \quad \Rightarrow \quad (m+3)(m-1) = 0$

$\Rightarrow m_1 = -3\,, \quad m_2 = 1$ real and distinct.

$$\therefore \; y_1 = e^{-3x}\,, \; y_2 = e^{x}$$

are the solutions, and $y = c_1 e^{-3x} + c_2 e^{x}$ is the general solution.

Example Solve the equation $(3D^3 + 5D^2 - 2D)y = 0$.

Solution:

The auxiliary equation $f(m) = 0 \Rightarrow \; 3m^3 + 5m^2 - 2m = 0 \; \Rightarrow m(3m^2 + 5m - 2) = 0$

$\Rightarrow \; m(m+2)(3m-1) = 0$

$\Rightarrow m_1 = 0\,, \quad m_2 = -2\,, \quad m_3 = \frac{1}{3}$ real and distinct.

The general solution (G.S.) is:

$$y = c_1 e^{0.x} + c_2 e^{-2x} + c_3 e^{\frac{1}{3}x} = \; c_1 + c_2 e^{-2x} + c_3 e^{\frac{1}{3}x}.$$

Exercises: Solve the following DE's:

1. $y''' + 3y'' - 4y' = 0$
2. $(D^2 - D - 2)y = 0$
3. $y'' - 4y = 0$
4. $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0.$

2. **Repeated Real Roots**:

For simplicity, and without loss of generality, consider a second homogeneous linear DE with constant coefficients

$$a_2 y'' + a_1 y' + a_0 y = 0$$

and the corresponding characteristic equation (auxiliary equation) is given by

$$a_2 m^2 + a_1 m + a_0 = 0.$$

And suppose it has repeated roots $m_1 = b\,, m_2 = b$, hence the two solutions $y_1 = e^{bx}\,, y_2 = e^{bx}$ which are linearly dependent, will not give the general solution (G.S.), since

$$y = c_1 y_1 + c_2 y_2 = (c_1 + c_2)e^{bx} = c_3 e^{bx}.$$

In order to make them independent solutions, we let

$$y_1 = e^{bx}, y_2 = xe^{bx}$$

Which are linearly independent. Therefore, the G.S. is given by

$$y = (c_1 + c_2 x)e^{bx}.$$

Hence for n-th order equation, if m_1 is a repeated roots of k times, and $m_{k+1}, m_{k+2}, \cdots, m_n$ are different (distinct) roots, then we let

$$y_1 = e^{m_1 x}, \quad y_2 = xe^{m_1 x}, \cdots, y_k = x^{k-1}e^{m_1 x}$$

and the other solutions will be

$$y_{k+1} = e^{(m_{k+1})x}, \; y_{k+2} = e^{(m_{k+2})x}, \cdots, y_n = e^{m_n x}.$$

Therefore, the G. S. is given by

$$y = (c_1 + c_2 x + \cdots + c_k x^{k-1})e^{m_1 x} + c_{k+1}e^{(m_{k+1})x} + \cdots + c_n e^{m_n x}.$$

Example Solve $y^{(4)} + 2y''' + y'' = 0.$

Solution: $f(D)y = 0, \; f(D) = D^4 + 2D^3 + D^2$

The auxiliary equation (characteristic equation) is given by:

$$m^4 + 2m^3 + m^2 = 0 \;\Rightarrow m^2(m^2 + 2m + 1) = 0$$

$$\Rightarrow \; m_1 = 0, \; m_2 = 0$$

$$(m^2 + 2m + 1) = 0 \;\Rightarrow (m + 1)(m + 1) = 0$$

$$\Rightarrow \; m_3 = -1, \; m_4 = -1.$$

$$\Rightarrow y_1 = e^{0x} = 1, \quad y_2 = xe^{0x} = x, \quad y_3 = e^{-x}, \quad y_4 = xe^{-x}$$

The general solution is given by

$$y = c_1 + c_2 x + (c_3 + c_4 x)e^{-x}.$$

Example Solve the equation $(D^5 - D^3)y = 0.$

Solution: The auxiliary (characteristic) equation is:

$$f(m) = 0 \quad \Rightarrow \; (m^5 - m^3) = 0$$

$$m^3(m^2 - 1) = 0$$

$$\Rightarrow \; m_1 = m_2 = m_3 = 0, \quad m_4 = 1, \quad m_5 = -1$$

$$y_1 = e^{0x} = 1, \quad y_2 = xe^{0x} = x, \quad y_3 = x^2 e^{0x} = x^2, \; y_4 = e^x, \; y_5 = e^{-x}$$

The G.S. is:

$$y = c_1 + c_2x + c_3x^2 + c_4e^x + c_5e^{-x}.$$

Exercises: Solve the following DE's:

1. $4y''' + 4y'' + y' = 0$
2. $y''' + 2y'' = 0\,, \quad y(0) = -3,\ y'(0) = 0,\ y''(0) = 12.$
3. $(D^2 + 4D + 4)y = 0.$
4. $f(D) = 0,$ where $f(D) = D^6 - D^4.$

3. Imaginary (Complex) Roots:

Consider the second order differential equation:

$$a_2y'' + a_1y' + a_0y = 0$$

and suppose that the characteristic equation has a complex root $m_1 = a + ib$

$$\rightarrow \quad m_2 = a - ib$$

which is the conjugate root to the characteristic equation.

Then $y_1 = e^{m_1x} = e^{(a+ib)x} = e^{ax}.e^{ibx}$

$$y_1 = e^{ax}(cosbx + i\ sinbx)$$

Similarly, $y_2 = e^{ax}.e^{-ibx} = e^{ax}\big(cos(-bx) + i\ sin(-bx)\big)$

$$y_2 = e^{ax}(cosbx - i\ sinbx)$$

Because: $(\mathbf{sin}(-\boldsymbol{x}) = -\boldsymbol{sinx}\,, \quad \mathbf{cos}(-\boldsymbol{x}) = \boldsymbol{cosx}).$

Therefore, the G.S. is

$$y = c_1y_1 + c_2y_2$$

$$= c_1e^{ax}(cosbx + i\ sinbx) + c_2e^{ax}(cosbx - i\ sinbx)$$

$$= e^{ax}[(c_1\, cosbx + ic_1\, sinbx) + (c_2 cosbx - ic_2\, sinbx)]$$

$$= e^{ax}[(c_1 + c_2)\, cosbx + i(c_1 - c_2)\, sinbx]$$

$$\boxed{y = e^{ax}(c_3\, cosbx + c_4\, sinbx).}$$

Example Solve $(D^2 + 4D + 9)y = 0$.

Solution: The auxiliary equation is: $f(m) = 0$

$$\Rightarrow (m^2 + 4m + 9) = 0$$

$$\Rightarrow m = \frac{-4 \mp \sqrt{16 - 36}}{2} = \frac{-4 \mp \sqrt{-20}}{2} = \frac{-4 \mp 2\sqrt{5}\, i}{2} = -2 \mp \sqrt{5}\, i$$

$$\Rightarrow m_1 = -2 + \sqrt{5}\, i\,, \qquad m_2 = -2 - \sqrt{5}\, i$$

$$\therefore\ y = e^{-2x}\left(c_1\, cos\sqrt{5}x + c_2\, sin\sqrt{5}x\right).$$

Example Solve the equation $f(D)y = 0$

where $f(D) = (D + 1)(D^2 - 4D + 13)$.

Solution: The auxiliary equation is

$$f(m) = 0 \ \Rightarrow\ (m + 1)(m^2 - 4m + 13) = 0$$

$$\therefore\ m_1 = -1\,, \qquad m^2 - 4m + 13 = 0 \ \Rightarrow\ m = \frac{4 \mp \sqrt{16 - 4 \times 13}}{2}$$

$$\therefore m = \frac{4 \mp \sqrt{-36}}{2} = \frac{4 \mp \sqrt{36\, i^2}}{2} = \frac{4 \mp 6i}{2} = 2 \mp 3i$$

$$\therefore\ m_1 = -1\ , \qquad m_2 = 2 + 3i\,, \quad m_3 = 2 - 3i.$$

∴ The G.S. is

$$y = c_1 e^{-x} + e^{2x}(c_2\, cos3x + c_3\, sin3x).$$

Example Find the G.S. of

$$\frac{d^2y}{dx^2} + 9y = 0.$$

Solution: $f(D)y = 0 \;\Rightarrow\; (D^2 + 9)y = 0.$

The auxiliary equation is $f(m) = 0,\;\; m^2 + 9 = 0$

$$\therefore \;\; m = \mp 3i \;\Rightarrow\; m_1 = 3i\,, \qquad m_2 = -3i.$$

The G.S. is: $\;y = e^{0x}(c_1\, cos3x + c_2\, sin3x).$

Example Solve

$$\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + 16y = 0. \qquad (1)$$

Solution: Define the operator $D = \frac{d}{dx}$

$\therefore$ eq. (1) can be written as

$$(D^4 + 8D^2 + 16)y = 0,$$

then the characteristic equation

$$f(m) = m^4 + 8m^2 + 16 = 0$$

$$(m^2 + 4)(m^2 + 4) = 0 \;\;\Rightarrow\;\; (m^2 + 4)^2 = 0$$

i.e. $\;m_1 = 2i\,,\; m_2 = -2i\,,\; m_3 = 2i\,,\; m_4 = -2i.$

Hence, the G.S. is given by

$$y = e^{0x}[c_1\, cos2x + c_2\, sin2x] + e^{0x}[c_3x\, cos2x + c_4x\, sin2x]$$

$$y = (c_1 + c_3x)\, cos2x + (c_2 + c_4x)\, sin2x.$$

Example Solve the equation

$$D(D-1)^3(D^2 + 1)(D^2 + 2D + 2)^2y = 0.$$

Solution: The auxiliary equation

$$f(m) = 0 \;\Rightarrow\; m(m-1)^3(m^2 + 1)(m^2 + 2m + 2)^2 = 0$$

$m_1 = 0,$

$(m-1)^3 = 0 \Rightarrow\; m_2 = m_3 = m_4 = 1.$

$m^2 + 1 = 0 \;\Rightarrow\; m^2 = -1 \;\Rightarrow\; m = \mp\, i\,, \qquad \therefore\; m_5 = i\,,\; m_6 = -i$

$(m^2+2m+2)^2=0 \;\Rightarrow\; (m^2+2m+2)\,(m^2+2m+2)=0$

$(m^2+2m+2)=0 \;\Rightarrow\; m=\frac{-2\mp\sqrt{4-8}}{2}=\frac{-2\mp\sqrt{-4}}{2}=\frac{-2\mp 2i}{2}=-1\mp i$

$\therefore\; m_7=m_8=-1+i\,,\; m_9=m_{10}=-1-i$

So, the roots are: $m=0,1,1,1,i,-i,(-1+i),\;(-1+i),\;(-1-i),(-1-i)$.

The G.S. is:

$$y=c_1e^{0x}+(c_2+c_3x+c_4x^2)e^x+e^{0x}(c_5\,cosx+c_6\,sinx)+e^{-x}[(c_7+c_8x)\,cosx+(c_9+c_{10}x)\,sinx].$$

Exercises: Solve the following DE's

1. $y''-4y'+7y=0.$
2. $\frac{d^2y}{dx^2}-9y=0.$

Chapter 4

Non-Homogeneous Linear Equations with Constant Coefficients

Chapter 4

Non-Homogeneous Linear Equations with Constant Coefficients:

Given a non-homogeneous linear differential equation with constant coefficients

$$(b_0D^n + b_1D^{n-1} + \cdots + b_{n-1}D + b_n)y = R(x)$$

Or

$$f(D)y = R(x) \qquad (1)$$

The G.S. of eq. (1) is $y = y_c + y_p$, where y_c is the complementary solution of eq. (1) which is also the G.S. of the homogeneous linear DE

$$(b_0D^n + b_1D^{n-1} + \cdots + b_{n-1}D + b_n)y = 0$$

Or

$$f(D)y = 0 \qquad (2)$$

and y_p is any particular solution of eq. (1).

Various methods for finding the y_p of eq. (1) are available where $b_0, b_1, \cdots, b_n$ are constants.

4.1 Solution of A Non-Homogeneous Equations:

There are several methods to find y_p, here are some of them:

I. Inspection Method:

Our aim is to find y_p of eq. (1) by inspection. There are three cases given next.

Case [I]: If $R(x) = R_0$ (R_0 is constant) and $b_n \neq 0$, that is eq. (1) has the form

$$(b_0D^n + b_1D^{n-1} + \cdots + b_{n-1}D + b_n)y = R_0 \qquad (3)$$

Then $y_p = \frac{R_0}{b_n}$.

Proof: If $y_p = \frac{R_0}{b_n}$ is a solution of (3), then

$$(b_0D^n + b_1D^{n-1} + \cdots + b_{n-1}D + b_n)y_p = (b_0D^n + b_1D^{n-1} + \cdots + b_{n-1}D + b_n)\frac{R_0}{b_n}$$

$$= b_0D^n\,\frac{R_0}{b_n} + b_1D^{n-1}\,\frac{R_0}{b_n} + \cdots + b_{n-1}D\,\frac{R_0}{b_n} + b_n\frac{R_0}{b_n} = 0 + 0 + \cdots + 0 + R_0 = R_0.$$

Example Solve the eq. $(D^2 - 3D + 2)y = 16$ by inspection method.

Solution: To find y_c , define the auxiliary equation $f(m) = 0$,

$$\Rightarrow\ f(m) = m^2 - 3m + 2 = 0 \ \Rightarrow\ (m-1)(m-2) = 0$$

$$\therefore\ m_1 = 1\,,\ \ m_2 = 2$$

$$\therefore\ y_c = c_1e^x + c_2e^{2x}.$$

To find y_p ,

$$R_0 = 16\,,\ b_0 = 2 \neq 0 \ \Rightarrow\ y_p = \frac{R_0}{b_2} = \frac{16}{2} = 8.$$

$\therefore$ The G.S. is: $y = y_c + y_p$

$$\Rightarrow\ y = c_1e^x + c_2e^{2x} + 8.$$

Case [II]: Given $f(D)y = R(x)$ (1)

If each term of $f(D)y$ is proportional to $R(x)$, then set $y_p = AR(x)$ and then find A by direct use of eq. (1).

Example Solve the equation $(D^2 + 4)y = sin3x$ (1)

Using inspection method.

Solution: to find y_c ,

$$f(m) = 0 \;\Rightarrow\; m^2 + 4 = 0 \;\Rightarrow\; m = \mp 2i\,; \quad m_1 = 2i\,, \qquad m_2 = -2i$$

$$\therefore \boxed{y_c = c_1\, cos2x + c_2\, sin2x.}$$

To find y_p

$R(x) = sin3x$ is proportional to each term of $(D^2 + 4)y$

Set $\boxed{y_p = A\, sin3x}$ (2)

$$y_p' = 3A\, cos3x, \quad y_p'' = -9A\, sin3x.$$

Substitute y_p , y_p'' in eq. (1) gives

$$-9A\, sin3x + 4A\, sin3x = \sin 3x$$

$$\therefore \quad -5A\, sin3x = sin3x \quad \Rightarrow \quad -5A = 1 \quad \Rightarrow \boxed{A = -\frac{1}{5}}$$

Put the value of A in (2) $\Rightarrow$ $y_p = -\frac{1}{5}\, sin3x.$

$\therefore$ The G.S. is: $\boxed{y = y_c + y_p = c_1\, cos2x + c_2\, sin2x - \frac{1}{5}\, sin3x.}$

Example Solve the equation $(D^2 + 3D - 4)y = 18\, e^{2x}$ (1)

by inspection method.

Solution: $f(m) = 0 \quad \Rightarrow \quad m^2 + 3m - 4 = 0 \quad \Rightarrow \quad (m-1)(m+4) = 0$

$$\therefore\; m_1 = 1\,, \quad m_2 = -4$$

$$\therefore \boxed{y_c = c_1\, e^{x} + c_2\, e^{-4x}\,.}$$

To find y_p

$R(x) = 18e^{2x}$, is proportional to each term in $(D^2 + 3D - 4)y$.

Set $y_p = Ae^{2x} \;\Rightarrow\; y_p' = 2Ae^{2x} \;\Rightarrow\; y_p'' = 4Ae^{2x}$.

Substitute y_p , y_p' , y_p'' in eq. (1), we have

$$4Ae^{2x} + 6Ae^{2x} - 4Ae^{2x} = 18e^{2x}$$

$$6Ae^{2x} = 18e^{2x}$$

By comparison $\Rightarrow \quad 6A = 18 \quad \Rightarrow \quad A = 3$

$$\therefore \quad y_p = 3e^{2x}.$$

$\therefore$ The G.S. is: $\boxed{y = y_c + y_p = c_1 e^{x} + c_2 e^{-4x} + 3e^{2x}.}$

Remark Case [II] of inspection method will save no work if there exists a term of $f(D)y$ is not proportional to $R(x)$. To see that, consider the following example.

Example Solve $(D^2 + 4D + 4)y = sin3x \qquad (1)$

Set $\quad y_p = A\, sin3x \quad \Rightarrow \quad y_p' = 3A\, cos3x \quad \Rightarrow \quad y_p'' = -9A\, sin3x.$

Substitute $\quad y_p\,,\ y_p'\,, y_p''$ in (1), gives

$$-9A\, sin3x + 12A\, cos3x + 4A\, sin3x = sin3x$$

$$-5A\, sin3x + 12A\, cos3x = sin3x$$

$\Rightarrow \quad -5A = 1 \quad$ and $\quad 12A = 0 \quad \Rightarrow \quad A = -\frac{1}{5} \quad$ and $\quad A = 0 \quad$ impossible.

This example shows that case [II] of inspection method will not provide a particular solution for eq. (1) because there is a term in $(D^2 + 4D + 4)y$ is not proportional to $sin3x$.

Case [III] Given $\quad f(D)y = R(x) \qquad (1)$

Where $R(x)$ may be split into parts, say

$$R(x) = R_1(x) + R_2(x) + \cdots + R_k(x).$$

To find a particular solution for eq. (1) by inspection, we treat separate terms of $R(x)$ independently. That is:

If $y_1, y_2, \cdots, y_k$ are particular solutions of $f(D)y = R_1(x), f(D)y = R_2(x), \cdots, f(D)y = R_k(x)$, respectively then

$$y_p = y_1 + y_2 + \cdots + y_k$$

is a particular solution of $f(D)y = R_1(x) + R_2(x) + \cdots + R_k(x) = R(x)$.

Example Solve the equation $(D^2-9)y = 3e^x + x - sin4x$ (1)

by inspection method.

Solution: The auxiliary equation $f(m) = 0 \Rightarrow m^2 - 9 = 0 \Rightarrow m_1 = 3,\ m_2 = -3$

$$\therefore \boxed{y_c = c_1 e^{3x} + c_2 e^{-3x}}$$

To find y_p:

$$R_1(x) = 3e^x,\ R_2(x) = x\ , R_3(x) = -sin4x.$$

Let y_1 be a P.S. of $(D^2-9)y = 3e^x$ (2)

y_2 be a P.S. of $(D^2-9)y = x$ (3)

y_3 be a P.S. of $(D^2-9)y = -sin4x$ (4)

and $y_p = y_1 + y_2 + y_3$.

Set $y_1 = Ae^x \Rightarrow y_1' = Ae^x\ ,\ y_1'' = Ae^x$.

Substitute y_1'' and y_1 in (2) gives

$$Ae^x - 9Ae^x = 3e^x \Rightarrow -8Ae^x = 3e^x$$

$$\therefore -8A = 3 \qquad \therefore A = -\frac{3}{8}$$

$$\therefore \boxed{y_1 = -\frac{3}{8}e^x.}$$

Set $y_2 = Bx \Rightarrow y_2' = B\ ,\ y_2'' = 0$.

Substitute y_2'' and y_2 in (3) gives

$$0 - 9Bx = x \Rightarrow -9B = 1$$

$$\therefore B = -\frac{1}{9}$$

$$\therefore \boxed{y_2 = -\frac{1}{9}x.}$$

Set $y_3 = C\ sin4x \Rightarrow y_3' = 4C\ cos4x\ ,\ y_3'' = -16C\ sin4x$.

Substitute y_3'' and y_3 in (4) gives

$$-16C\ sin4x - 9C\ sin4x = -sin4x$$

$$\Rightarrow \quad -25C\ sin4x = -sin4x$$

$$\therefore\ -25C = -1 \quad \Rightarrow \quad C = \frac{1}{25}$$

$$\therefore \quad \boxed{y_3 = \frac{1}{25} sin4x.}$$

$$\therefore \quad y_p = y_1 + y_2 + y_3 = -\frac{3}{8}e^x - \frac{1}{9}x + \frac{1}{25}sin4x$$

$\therefore$ The G.S. of eq. (1) is

$$y = y_c + y_p = c_1 e^{3x} + c_2 e^{-3x} - \frac{3}{8}e^x - \frac{1}{9}x + \frac{1}{25}sin4x.$$

Example Solve the equation $(D^2 + 4)y = sinx + sin2x$ (1)

Solution: $f(m) = 0 \quad \Rightarrow \quad m^2 + 4 = 0 \quad \Rightarrow \quad m = \mp 2i \quad \therefore \quad m_1 = 2i\ ,\ m_2 = -2i$

$$\therefore \quad \boxed{y_c = c_1\ cos2x + c_2\ sin2x.}$$

To find y_p:

$$R_1(x) = sinx\ , \quad R_2(x) = sin2x.$$

Let y_1 be a P.S. of $(D^2 + 4)y = sinx$ (2)

y_2 be a P.S. of $(D^2 + 4)y = sin2x$ (3)

and $y_p = y_1 + y_2$.

Set $y_1 = A\ sinx \ \Rightarrow\ y_1' = A\ cosx\ ,\ y_1'' = -A\ sinx.$

Put y_1'' and y_1 in (2) gives

$$-A\ sinx + 4A\ sinx = sinx \quad \Rightarrow \quad 3A\ sinx = sinx$$

$$\Rightarrow \quad A = \frac{1}{3} \qquad \therefore \quad \boxed{y_1 = \frac{1}{3}\ sinx}$$

Set $y_2 = B\ sin2x \ \Rightarrow\ y_2' = 2B\ cos2x\ ,\ y_2'' = -4B\ sin2x.$

Put y_2'' and y_2 in (3) gives

$$-4B\ sin2x + 4B\ sin2x = sin2x \quad \Rightarrow \quad 0 = sin2x$$

∴ No particular solution can be found by inspection. So, using the later method [Method of Undetermined Coefficients], we shall solve for P.S. for

$$(D^2 + 4)y = sin2x \qquad \textbf{[H. W. for you]!}$$

$$\Rightarrow \boxed{y_2 = -\frac{1}{4}x\ cos2x.}$$

$$\therefore \quad y_p = y_1 + y_2 = \frac{1}{3}\ sinx - \frac{1}{4}x\ cos2x$$

The G.S. of eq. (1) is

$$y = y_c + y_p = c_1\ cos2x + c_2\ sin2x + \frac{1}{3}\ sinx - \frac{1}{4}x\ cos2x.$$

Exercises Use inspection method to solve the following

1. $(D^2 + 4)y = 12.$
2. $(D^2 + 4)y = 6\ sinx.$
3. $(D^2 + 4)y = 8x + 1 - 15e^x.$
4. $(D^2 + 1)y = e^x + 3x.$
5. $(D^3 - 1)y = e^{-x}.$

II. Undetermined Coefficients Method

Summary

Non-homogeneity	Guess
e^{kt}	Ae^{kt}
$(sinkt)$ or $(coskt)$ or $(sinkt + coskt)$	$A\ coskt + B\ sinkt$
Degree n polynomial	$A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0$

Let we have the following linear, non-homogeneous n-th order ODE's with constant coefficients, $f(D)y = R(x)$, for example:

1. $f(D)y = x^n$, then set y_p as a polynomial of degree n.

For example: if $f(D)y = x^3$

Then set $y_p = Ax^3 + Bx^2 + CX + D$.

2. $f(D)y = \begin{matrix} sinkx \\ coskx \end{matrix}$,
   Then set $y_p = A\ coskx + B\ sinkx$.

3. $f(D)y = cosx + sinx$,
   Then set $y_p = A\ cosx + B\ sinx$.

4. $f(D)y = x^3 + x^2 + X$,
   Then set $y_p = Ax^3 + Bx^2 + CX + D$.

5. $f(D)y = 3x + 7$,
   Then set $y_p = AX + B$.

6. $f(D)y = e^{kx}$, $k \in R$
   Then set $y_p = Ae^{kx}$.

7. $f(D)y = e^x\ cosx$
   Then set $y_p = e^x(A\ cosx + B\ sinx) = Ae^x\ cosx + Be^x\ sinx$.

8. $f(D)y = e^{2x}\,x^2$
   Then set $y_p = e^{2x}(A\,x^2 + Bx + C) = Ae^{2x}\,x^2 + Be^{2x}\,x + Ce^{2x}$.

9. $f(D)y = xe^x + sin2x$
   Then set $y_p = e^x(Ax + B) + C\,cos2x + D\,sin2x$
   $= Axe^x + Be^x + C\,cos2x + D\,sin2x$.

10. $f(D)y = x^3 + sin3x + 4e^{3x}$
    Then set $y_p = A\,x^3 + Bx^2 + Cx + D + E\,cos3x + F\,sin3x + G\,e^{3x}$.

11. $f(D)y = 5x + e^x$
    Then set $y_p = A\,x + B + Ce^x$.

Remark In contrast to inspection method, if $R(x) = R_1(x) + R_2(x) + \cdots + R_k(x)$, then set y_p as a one expression, not separate each term $R_i(x), i = 1, \cdots, k$ independently.

Example Solve $y'' + y' - 2y = 2x - 40\,cos2x$ (1)

Solution: To find the complementary function, consider

$$y'' + y' - 2y = 0 \quad \Rightarrow \quad f(m) = m^2 + m - 2 = 0$$

$$\Rightarrow \quad (m+2)(m-1) = 0 \qquad \therefore \quad m_1 = -2\,, \quad m_2 = 1.$$

$$\Rightarrow \quad y_1 = e^{-2x}\,, \quad y_2 = e^x$$

$$\therefore \quad \boxed{y_c = c_1\,e^{-2x} + c_2\,e^x.}$$

Now, to find y_p:

Set $y_p = Ax + B + C\,cos2x + D\,sin2x$

To find A, B, C, and D:

$$y_p' = A - 2C\,sin2x + 2D\,cos2x$$

$$y_p'' = -4C\,cos2x - 4D\,sin2x$$

Put y_p, y_p', y_p'' in eq. (1)

$$\Rightarrow \quad -4C\,cos2x - 4D\,sin2x + A - 2C\,sin2x + 2D\,cos2x$$

$$- 2(Ax + B + C\,cos2x + D\,sin2x) = 2x - 40\,cos2x$$

$$\Rightarrow \quad -4C\ cos2x - 4D\ sin2x + A - 2C\ sin2x + 2D\ cos2x - 2Ax - 2B - 2C\ cos2x - 2D\ sin2x = 2x - 40\ cos2x$$

$$\Rightarrow \quad (-4C + 2D - 2C)\ cos2x + (-4D - 2C - 2D)\ sin2x - 2Ax + (A - 2B) = 2x - 40\ cos2x$$

$$\Rightarrow \quad (-6C + 2D)\ cos2x + (-6D - 2C)\ sin2x - 2Ax + (A - 2B) = 2x - 40\ cos2x$$

Hence,

$$\left.\begin{aligned} -6C + 2D &= -40 \\ -6D - 2C &= 0 \end{aligned}\right\} \rightarrow \boxed{C = 6,\ D = -2}$$

$$-2A = 2 \quad \Rightarrow \quad \boxed{A = -1}$$

$$A - 2B = 0 \quad \Rightarrow \quad -1 - 2B = 0 \quad \Rightarrow \quad \boxed{B = -\frac{1}{2}}$$

$$\therefore \boxed{y_p = -x - \frac{1}{2} + 6\ cos2x - 2\ sin2x.}$$

Hence, the general solution is given by

$$y = y_c + y_p = c_1\ e^{-2x} + c_2\ e^{x} - x - \frac{1}{2} + 6\ cos2x - 2\ sin2x.$$

Example Find the particular solution of $y''' - y' = 4e^{-x} + 3e^{2x}$ (1)
where $x = 0,\ y = 0,\ y' = -1,\ y'' = 2.$

Solution: To find the complementary solution y_c ,
Consider $y''' - y' = 0$

$$\Rightarrow \ f(m) = m^3 - m = 0 \ \Rightarrow \ m(m^2 - 1) = 0 \ \Rightarrow \ m_1 = 0$$

$$(m^2 - 1) = 0 \ \Rightarrow \ m_2 = 1\ ,\ m_3 - 1$$

$$\therefore \ y_1 = e^{0x} = 1\ ,\ y_2 = e^{x}\ ,\ y_3 = e^{-x}$$

$$\Rightarrow \ \boxed{y_c = c_1 + c_2 e^{x} + c_3 e^{-x}.}$$

Now, to find the particular solution y_p:
Set $y_p = Axe^{-x} + Be^{2x}$
To find A and B ,

$$y_p' = Ae^{-x} - Axe^{-x} + 2Be^{2x}$$

$$y_p'' = -Ae^{-x} - Ae^{-x} + Axe^{-x} + 4Be^{2x}$$

$$y_p''' = Ae^{-x} + Ae^{-x} + Ae^{-x} - Axe^{-x} + 8Be^{2x}$$

$$= 3Ae^{-x} - Axe^{-x} + 8Be^{2x}$$

Put y_p''' and y_p' in eq. (1)

$$\Rightarrow \quad 3Ae^{-x} - Axe^{-x} + 8Be^{2x} - Ae^{-x} + Axe^{-x} - 2Be^{2x} = 4e^{-x} + 3e^{2x}$$

$$\Rightarrow \quad 2Ae^{-x} + 6Be^{2x} = 4e^{-x} + 3e^{2x}$$

$$\Rightarrow \quad 2A = 4 \quad \Rightarrow \quad \boxed{A = 2} \text{ and } \quad 6B = 3 \quad \Rightarrow \quad \boxed{B = \frac{1}{2}}$$

$$\therefore \boxed{y_p = 2xe^{-x} + \frac{1}{2}e^{2x}.}$$

$\therefore$ The general solution is given by:

$$\boxed{y = y_c + y_p = c_1 + c_2e^{x} + c_3e^{-x} + 2xe^{-x} + \frac{1}{2}e^{2x}.}$$

This is the general solution (G.S.) of eq.(1).

Now, we have to determine c_1, c_2, c_3 according to the conditions:

$x = 0,\ y = 0,\ y' = -1,\ y'' = 2.$

$$y = c_1 + c_2e^{x} + c_3e^{-x} + 2xe^{-x} + \frac{1}{2}e^{2x} \xrightarrow{x=0,\ y=0} 0 = c_1 + c_2 + c_3 + \frac{1}{2} \qquad (2)$$

$$y' = c_2e^{x} - c_3e^{-x} - 2xe^{-x} + 2e^{-x} + e^{2x} \xrightarrow{x=0,\ y'=-1} -1 = c_2 - c_3 + 2 + 1 \qquad (3)$$

$$y'' = c_2e^{x} + c_3e^{-x} + 2xe^{-x} - 2e^{-x} - 2e^{-x} + 2e^{2x}$$

$$\xrightarrow{x=0,\ y''=2} \quad 2 = c_2 + c_3 - 2 - 2 + 2 \qquad (4)$$

From eq. (2), eq. (3), and eq. (4)

$$\Rightarrow \quad c_1 = -\frac{9}{2}, \quad c_2 = 0, \quad c_3 = 4.$$

$\therefore$ The P.S. of (1) is

$$\boxed{y = -\frac{9}{2} + 4e^{-x} + 2xe^{-x} + \frac{1}{2}e^{2x}.}$$

Example Find the G.S. of the equation $(D^2 + 1)y = sinx$ (1)

Solution: To find y_c: we let $(D^2 + 1)y = 0$

$\therefore$ The auxiliary equation $f(m) = 0$ has roots $m_1 = i\ ,\ m_2 = -i$

$$\therefore\ y_c = c_1\ cosx + c_2\ sinx.$$

Now, to find y_p:

Set $y_p = Ax\ cosx + Bx\ sinx$

To find A and B:

$$y_p' = A(-x\ sinx + cosx) + B(x\ cosx + sinx)$$

$$y_p'' = A(-x\ cosx - 2sinx) + B(-x\ sinx + 2cosx)$$

Substitute y_p'' and y_p in (1), we get

$$\Rightarrow\quad A(-x\ cosx - 2sinx) + B(-x\ sinx + 2cosx) + Ax\ cosx + Bx\ sinx = sinx$$

$$\Rightarrow\quad -Ax\ cosx - 2A\ sinx - Bx\ sinx + 2B\ cosx + Ax\ cosx + Bx\ sinx = sinx$$

$$\Rightarrow\quad -2A\ sinx + 2B\ cosx = sinx$$

$$\therefore\quad -2A = 1\ ,\qquad -2B = 0\qquad \Rightarrow\quad A = -\frac{1}{2}\ ,\qquad B = 0.$$

$$\therefore\quad y_p = -\frac{1}{2}\ x\ cosx.$$

$\therefore$ The G.S. of (1) is:

$$y = y_c + y_p = c_1\ cosx + c_2\ sinx - \frac{1}{2}\ x\ cosx.$$

Exercises Find the solution of:

1. $(D^2 - 1)y = 8xe^x$
2. $(D^2 - 3D + 2)y = 2x^3 - 9x^2 + 6x$
3. $(D^4 - 1)y = 7x^2$
4. $(D^3 - D^2 + D - 1)y = 4sinx$
5. $y'' + 6y' + 13y = 60\ cosx + 26$
6. $y'' + 3y' + 2y = 1 + 3x + x^2$
7. $y'' + y' = -cosx$
8. $y'' - y = e^{-x}(2sinx + 4cosx)$
9. $y''' - y' = x$
10. $y''' + y'' - 4y' - 4y = 3e^{-3x} - 4x - 6.$

III. Solution By Variation of Parameter

For simplicity consider a second-order DE

$$a_2y'' + a_1y' + a_0y = f(x) \qquad (1)$$

In order to solve eq. (1), consider the homogeneous differential equation associated with (1) which is

$$a_2y'' + a_1y' + a_0y = 0 \qquad (2)$$

and let y_1 and y_2 are the linearly independent solutions of (2). Now, to find the solution of (1). Let $y_p = A(x)y_1 + B(x)y_2$ be any particular solution of (1), hence

$$y_p' = A(x)y_1' + A'(x)y_1 + B(x)y_2' + B'(x)y_2$$

in order to get a simplified equation for y_p'', we suppose that

$$A'(x)y_1 + B'(x)y_2 = 0 \qquad (3)$$

and therefore $y_p'' = A(x)y_1'' + B(x)y_2'' + A'(x)y_1' + B'(x)y_2'$

Now, substitute in the original eq. (1)

$$\rightarrow \quad a_2y_p'' + a_1y_p' + a_0y_p = f(x)$$

$$\rightarrow \quad a_2(Ay_1'' + By_2'' + A'y_1' + B'y_2') + a_1(Ay_1' + By_2') + a_0(Ay_1 + By_2) = f(x)$$

$$\rightarrow \quad A(a_2y_1'' + a_1y_1' + a_0y_1) + B(a_2y_2'' + a_1y_2' + a_0y_2) + a_2(A'y_1' + B'y_2') = f(x)$$

Since y_1 and y_2 are solutions $\rightarrow$ $a_2y_1'' + a_1y_1' + a_0y_1 = 0$, $\quad a_2y_2'' + a_1y_2' + a_0y_2 = 0$

$$\therefore \quad a_2(A'y_1' + B'y_2') = f(x)$$

$$\therefore \; A'y_1' + B'y_2' = \frac{f(x)}{a_2} \qquad (4)$$

Hence, solving (3) and (4) which are

$$A'(x)y_1 + B'(x)y_2 = 0 \;\;] * y_2'$$

$$A'(x)\,y_1' + B'(x)\,y_2' = \frac{f(x)}{a_2} \;\;] * y_2$$

$$\rightarrow \quad A'(x)y_1y_2' + B'(x)y_2y_2' = 0 \qquad (5)$$

$$\mp A'(x)\,y_1'y_2 \mp B'(x)\,y_2'\,y_2 = \mp\frac{f(x)}{a_2}.y_2$$

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$$\xrightarrow{yields} \quad A'(x)[y_1y_2' - y_1'y_2] = \frac{-f(x)}{a_2}.y_2$$

$$\therefore \quad \boxed{A'(x) = \frac{-f(x)\,y_2}{a_2\,(y_1y_2' - y_1'y_2)}}. \quad \text{Put in (5)}$$

$$\rightarrow \quad \frac{-f(x)\,y_2\,y_1\,y_2'}{a_2\,(y_1y_2' - y_1'y_2)} + B'(x)y_2y_2' = 0$$

$$\therefore \quad B'(x) = \frac{f(x)y_1\ y_2y_2'}{a_2\,(y_1y_2' - y_1'y_2).y_2y_2'} = \boxed{\frac{f(x)y_1}{a_2\,(y_1y_2' - y_1'y_2)}}$$

Then by direct integration we find $A(x)$ and $B(x)$. So, the desired P.S. of eq. (1) is obtained.

Example: Solve the non-homogeneous equation $y'' + y = tanx$.

Solution: The homogeneous equation is $y'' + y = 0 \quad \rightarrow \quad f(m) = 0$

$$m^2 + 1 = 0 \quad \rightarrow \quad m^2 = -1 \quad \rightarrow m_1 = -i\,, \qquad m_2 = i\,.$$

$$\therefore \quad y_1 = cosx\,, \quad y_2 = sinx$$

$$\therefore \quad y_c = c_1cosx + c_2sinx$$

Now, to find the particular solution of $y'' + y = tanx$, let

$$y_p = A(x)y_1 + B(x)y_2 = A\,cosx + B\,sinx.$$

$$\therefore \quad A'(x) = \frac{-\,y_2\,f(x)}{a_2\,(y_1y_2' - y_2y_1')} = \frac{-sinx\,tanx}{1} = -sinx\,tanx$$

$$= \frac{-sin^2x}{cosx} = \frac{(cos^2x - 1)}{cosx} = cosx - secx$$

$$\therefore \quad A(x) = \int (cosx - secx)dx = sinx - \ln|secx + tanx|$$

also $B'(x) = \frac{y_1\,f(x)}{1} = \frac{cosx\,tanx}{1} = cosx\,tanx = sinx$

$$\therefore \quad B(x) = \int sinx\,dx = -cosx$$

$$\therefore \quad y_p = Ay_1 + By_2 = cosx\,sinx - cosx\ln|secx + tanx| - sinx\,cosx$$

and the general solution is $y = y_c + y_p$.

Example: Solve $y''' - 6y'' + 11y' - 6y = e^x$.

Solution: The solution of $y''' - 6y'' + 11y' - 6y = 0$ is given by

$$y_c = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$$

Since $m_1 = 1$, $m_2 = 2$, $m_3 = 3$

assume that the particular solution is given by:

$$y_p = Ae^x + Be^{2x} + Ce^{3x}$$

$$y_p' = Ae^x + 2Be^{2x} + 3Ce^{3x} + A'e^x + B'e^{2x} + C'e^{3x}$$

With the condition that

$$A'e^x + B'e^{2x} + C'e^{3x} = 0$$

$$\therefore \quad y_p' = Ae^x + 2Be^{2x} + 3Ce^{3x}$$

also, $y_p'' = Ae^x + 4Be^{2x} + 9Ce^{3x} + A'e^x + 2B'e^{2x} + 3C'e^{3x}$

and also the condition

$$A'e^x + 2B'e^{2x} + 3C'e^{3x} = 0$$

Which leaves, $y_p'' = Ae^x + 4Be^{2x} + 9Ce^{3x}$

Similarly, $y_p''' = Ae^x + 8Be^{2x} + 27Ce^{3x} + A'e^x + 4B'e^{2x} + 9C'e^{3x}$

and substitute in the non-homogeneous equation, we have

$$A'e^x + 4B'e^{2x} + 9C'e^{3x} = e^x$$

Therefore, we have to solve

$$A'e^x + B'e^{2x} + C'e^{3x} = 0$$

$$A'e^x + 2B'e^{2x} + 3C'e^{3x} = 0$$

$$A'e^x + 4B'e^{2x} + 9C'e^{3x} = e^x$$

We have $A' = \frac{1}{2}$, $B' = -e^{-x}$, $C' = \frac{1}{2}e^{-3x}$, hence,

$$A = \int \frac{1}{2}\, dx = \frac{x}{2}, \qquad B = -\int e^{-x}\, dx = e^{-x}, \qquad C = \int \frac{1}{2} e^{-3x}\, dx = \frac{-1}{6}\, e^{-3x}.$$

$$\therefore \quad y_p = Ay_1 + By_2 + Cy_3$$

$$= \frac{x}{2}e^x + e^{-x}e^{2x} - \frac{1}{6}e^{-3x}e^{3x} = \frac{x}{2}e^x + e^x - \frac{1}{6}$$

$\therefore$ The general solution is given by

$$y = y_c + y_p = c_1e^x + c_2e^{2x} + c_3e^{3x} + \frac{x}{2}e^x + e^x - \frac{1}{6}.$$

Example: Solve the equation: $(D^2 - 3D + 2)y = (1 + e^{-x})^{-1}$ (1)

by variation of parameters method.

Solution: $f(m) = 0 \quad \rightarrow \quad m^2 - 3m + 2 = 0 \quad \rightarrow \quad m_1 = 1\,,\ m_2 = 2$

$$\rightarrow \quad \boxed{y_c = c_1e^x + c_2e^{2x}}$$

let $y_p = A(x)y_1 + B(x)y_2$

$$\boxed{y_p = A(x)e^x + B(x)e^{2x}} \qquad (2)$$

$$\therefore \quad A'(x) = \frac{-f(x)\,y_2}{a_2\,(y_1y_2' - y_2y_1')} = \frac{-e^{2x}}{(1 + e^{-x})(2e^xe^{2x} - e^{2x}e^x)} = \frac{-e^{2x}}{(1 + e^{-x}).e^{3x}}$$

$$\rightarrow \quad A'(x) = \frac{-e^{-x}}{1 + e^{-x}}$$

$$\therefore \quad A(x) = \int \frac{-e^{-x}}{1 + e^{-x}}\,dx \quad \rightarrow \quad \boxed{A(x) = \ln(1 + e^{-x})}$$

$$B'(x) = \frac{f(x)\,y_1}{a_2\,(y_1y_2' - y_2y_1')} = \frac{e^x}{(1 + e^{-x})(2e^xe^{2x} - e^{2x}e^x)} = \frac{e^x}{(1 + e^{-x}).e^{3x}}$$

$$\rightarrow \quad B'(x) = \frac{e^{-2x}}{1 + e^{-x}}$$

$$\therefore \quad B(x) = \int \frac{e^{-2x}}{1 + e^{-x}}\,dx \quad \rightarrow \quad B(x) = \int \left(e^{-x} - \frac{e^{-x}}{1 + e^{-x}}\right) dx$$

$$\xrightarrow{yields} \quad \boxed{B(x) = -e^{-x} + \ln(1 + e^{-x})}$$

Put the values of $A(x)$ and $B(x)$ in (2)

$$\xrightarrow{yields} \quad y_p = e^x \ln(1 + e^{-x}) + e^{2x}[-e^{-x} + \ln(1 + e^{-x})]$$

$$= e^x \ln(1 + e^{-x}) - e^x + e^{2x} \ln(1 + e^{-x})$$

$$= (e^x + e^{2x}) \ln(1 + e^{-x}) - e^x$$

∴ The G.S. of eq. (1) is

$$y = y_c + y_p = c_1 e^x + c_2 e^{2x} + (e^x + e^{2x}) \ln(1 + e^{-x}) - e^x$$

$$= (c_1 - 1)e^x + c_2 e^{2x} + (e^x + e^{2x}) \ln(1 + e^{-x}).$$

$$\therefore \quad \boxed{y = c_3 e^x + c_2 e^{2x} + (e^x + e^{2x}) \ln(1 + e^{-x})}.$$

Example: If x and e^x are independent solutions of the homogeneous equation associated with the non-homogeneous equation

$$(1 - x)y'' + xy' - y = 2(x - 1)^2 e^{-x} \qquad (1)$$

Use this fact and find the G.S. of the nonhomogeneous equation.

Solution: $\boxed{y_c = c_1 x + c_2 e^x}$

Let $\boxed{y_p = A(x)x + B(x)e^x}$ (2)

We have

$$y_1 = x\,, \quad y_2 = e^x\,, \quad f(x) = 2(x - 1)^2 e^{-x}$$

$$A'(x) = \frac{-f(x)\, y_2}{a_2\,(y_1 y_2' - y_2 y_1')} = \frac{-2(x-1)^2}{(1-x)(xe^x - e^x)} = \frac{-2(x-1)^2}{(1-x)(x-1)e^x}$$

$$A'(x) = \frac{-2(x-1)}{(1-x)e^x} * \frac{-1}{-1}$$

$$\rightarrow \quad A'(x) = 2e^{-x}.$$

$$\therefore \quad A(x) = \int 2e^{-x}\, dx \quad \rightarrow \quad \boxed{A(x) = -2e^{-x}}.$$

$$B'(x) = \frac{f(x)\,y_1}{a_2\,(y_1y_2' - y_2y_1')} = \frac{2(x-1)^2xe^{-x}}{(1-x)(xe^x - e^x)} = \frac{2x\,e^{-x}\,(x-1)^2}{(1-x)(x-1)e^x}\Bigg] * \frac{-1}{-1}$$

$$B'(x) = \frac{-2x\,e^{-x}\,(x-1)^2}{(x-1)(x-1)e^x} = \frac{-2x\,e^{-x}}{e^x}$$

$$\rightarrow \quad B'(x) = -2xe^{-2x}.$$

$$\therefore \; B(x) = \int -2xe^{-2x}\,dx \quad \rightarrow \quad B(x) = xe^{-2x} + \frac{1}{2}e^{-2x}$$

$$\xrightarrow{yields} \boxed{B(x) = \left(x + \frac{1}{2}\right)e^{-2x}.}$$

Put the values of $A(x)$ and $B(x)$ in (2)

$$\Rightarrow \quad y_p = -2xe^{-x} + \left(x + \frac{1}{2}\right)e^{-x} = \left(\frac{1}{2} + x - 2x\right)e^{-x} = \left(\frac{1}{2} - x\right)e^{-x}.$$

$\therefore$ The G.S. of eq. (1) is

$$\boxed{y = y_c + y_p = c_1x + c_2e^x + \left(\frac{1}{2} - x\right)e^{-x}.}$$

Exercises: Solve the following DE's using variation of parameter method

1. $y'' - 3y' + 2y = \frac{1}{1+e^{-x}}$
2. $y'' + y = secx\,tanx$
3. $y''' - y' = x$
4. $y''' + y' = tanx$
5. $y''' - 3y'' - y' + 3y = x^2e^x$
6. $(D^2 + 2D + 2)y = e^{-x}\,cscx$
7. $(D^2 - 3D + 2)y = \frac{e^{2x}}{(1+e^{2x})}$
8. $(D-1)(D-2)(D-3)y = e^x.$

IV. Solution By Reduction of Order: Second Order Linear DE's

The general form of the second order linear DE is:

$$y'' + r(x)y' + s(x)y = R(x) \qquad (1)$$

or $(D^2 + r(x)D + s(x))y = R(x)$.

Suppose we know that $y = y_1$ is a solution of the homogeneous equation

$$y'' + r(x)y' + s(x)y = 0 \qquad (2)$$

then $y_1'' + r(x)y_1' + s(x)y_1 = 0$.

In this method, we introduce a new dependent variable v such that $y = y_1 v$ will lead to a solution of equation (1)

$$y' = y_1 v' + y_1' v$$

$$y'' = y_1 v'' + 2y_1' v' + y_1'' v$$

Substitute y, y' and y'' in (1), we get

$$y_1 v'' + 2y_1' v' + y_1'' v + r(x)y_1 v' + r(x)y_1' v + s(x)y_1 v = R(x)$$

$$(y_1'' + r(x)y_1' + s(x)y_1)v + y_1 v'' + (2y_1' + r(x)y_1)v' = R(x)$$

$$0 + y_1 v'' + (2y_1' + r(x)y_1)v' = R(x) \qquad (3)$$

Set $w = v' \quad \rightarrow \quad w' = v''$ then eq. (3) become

$$y_1 w' + (2y_1' + r(x)y_1)w = R(x)$$

or

$$w' + \left(\frac{2y_1' + r(x)y_1}{y_1}\right)w = \frac{R(x)}{y_1}$$

or

$$\frac{dw}{dx} + P(x)w = Q(x) \qquad (4)$$

where $P(x) = \frac{2y_1' + r(x)y_1}{y_1}$ and $Q(x) = \frac{R(x)}{y_1}$

eq. (4) is the first-order linear DE and the solution is given by method [5] of Chapter2. Eq.(4) gives w, but $v' = \frac{dv}{dx} = w \quad \rightarrow \quad dv = w\,dx \quad \rightarrow \quad v = \int w\,dx$.

Then the G.S. of eq.(1) is $y = vy_1$.

Example: Solve the second order eq. $y'' - y = e^x$ by reduction of order method.

Solution: $y'' - y = e^x$ (1)

or $(D^2 - 1)y = e^x$

$$f(m) = 0 \quad \rightarrow \quad m^2 - 1 = 0 \quad \rightarrow \quad m = 1, -1$$

$$\therefore \quad y_c = c_1 e^x + c_2 e^{-x}$$

We have two independent solutions e^x and e^{-x} for the homogeneous equation $y'' - y = 0$.

Let $y = ve^x$

$$y' = ve^x + v'e^x$$

$$y'' = ve^x + 2v'e^x + v''e^x$$

Put y, y' and y'' in (1), gives

$$ve^x + 2v'e^x + v''e^x - ve^x = e^x \quad \rightarrow \quad (2v' + v'')e^x = e^x$$

$$v'' + 2v' = 1 \qquad (2)$$

Let $w = v' \rightarrow w' = v''$, then eq.(2) becomes

$w' + 2w = 1$ or $\frac{dw}{dx} + 2w = 1$ (3)

Eq.(3) is the first-order linear DE with $P(x) = 2$ and $Q(x) = 1$.

$$u(x) = e^{\int P(x)dx} = e^{\int 2dx} = e^{2x}.$$

The G.S. of eq.(3) is

$$w\,u(x) = \int u(x)Q(x)dx$$

$$\rightarrow \quad we^{2x} = \int e^{2x}dx = \frac{1}{2}e^{2x} + c$$

$$w = \frac{1}{2} + ce^{-2x}$$

But $w = v' = \frac{dv}{dx}$

$$\therefore \quad \frac{dv}{dx} = \frac{1}{2} + ce^{-2x} \quad \rightarrow \quad dv = \left(\frac{1}{2} + ce^{-2x}\right)dx$$

$$\rightarrow \quad \boldsymbol{v} = \int \left(\frac{1}{2} + ce^{-2x}\right)dx \quad \rightarrow \quad v = \frac{x}{2} - \frac{c}{2}e^{-2x} + c_2$$

$\therefore \quad v = \frac{x}{2} + c_1 e^{-2x} + c_2$ where $c_1 = -\frac{c}{2}$.

∴ The G.S. of eq. (1) is:

$$y = ve^x = \left(\frac{x}{2} + c_1 e^{-2x} + c_2\right)e^x$$

Or

$$y = c_1 e^{-x} + c_2 e^x + \frac{1}{2}xe^x.$$

Example: Solve the eq. $y'' + y = cscx$ by reduction of order.

Solution: $(D^2 + 1)y = cscx$ or $y'' + y = cscx = \frac{1}{sinx}$ (1)

$$f(m) = 0 \quad \rightarrow \quad m^2 + 1 = 0 \quad \rightarrow \quad m = \pm i$$

$$\therefore \quad y_c = c_1\, cosx + c_2\, sinx.$$

We have two independent solutions $cosx$ and $sinx$ for the homogeneous eq. $(D^2 + 1)y = 0$.

Let

$$\left.\begin{array}{c} y = v\, sinx \\ y' = v\, cosx + v'\, sinx \\ y'' = -v\, sinx + 2v'\, cosx + v''\, sinx. \end{array}\right] \qquad (2)$$

Put (2) in (1) $\rightarrow \quad -v\, sinx + 2v'\, cosx + v''\, sinx + v\, sinx = cscx$

$$\rightarrow \quad v''\, sinx + 2v'\, cosx = cscx \quad \rightarrow \quad v'' + (2\, cotx)v' = csc^2 x \qquad (3)$$

Let $w = v' \quad \rightarrow \quad w' = v''$ then eq. (3) becomes

$$w' + (2\, cotx)w = csc^2 x$$

or $$\frac{dw}{dx} + (2\, cotx)w = csc^2 x \qquad (4)$$

eq. (4) is the first-order linear DE with $P(x) = 2\,cotx$ and $Q(x) = csc^2x$.

Then $u(x) = e^{\int P(x)dx} = e^{\int 2\,cotx\,dx} = e^{2\ln(sinx)} = e^{\ln(sin^2x)} = sin^2x$.

The G. S. of eq.(4) is:

$$w\,u(x) = \int u(x)Q(x)dx \quad \rightarrow \quad w\,sin^2x = \int sin^2x.csc^2x\,dx \quad \rightarrow \quad w\,sin^2x = \int dx$$

$$\rightarrow \quad w\,sin^2x = x + c$$

$$\therefore \quad w = (x + c)\,csc^2x.$$

But $w = v' = \frac{dv}{dx}$

$$\therefore \quad \frac{dv}{dx} = (x + c)\,csc^2x \quad \rightarrow \quad dv = (x + c)csc^2x\;\;dx$$

$$\therefore \quad v = \int (x + c)\,csc^2x\;\;dx$$

$$\rightarrow \quad v = -(x + c)\,cotx + \ln(sinx) + c_2$$

The G.S. of eq. (1) is

$$y = v\,sinx = [-(x + c)\,cotx + ln(sinx) + c_2]\,sinx.$$

$$= -(x + c)\,cosx + sinx\;\ln(sinx) + c_2\,sinx$$

$$= -x\,cosx - c\,cosx + sinx\;\ln(sinx) + c_2\,sinx$$

$\therefore$ $y = c_1\,cosx + c_2\;\;sinx + sinx\;\ln(sinx) - x\,cosx,$ where $c_1 = -c$.

Example: Solve the eq. $(D^2 - 5D + 6)y = 2e^x$ by reduction of order method.

Solution: We have $f(D)y = (D^2 - 5D + 6)y = 2e^x$ (1)

$$f(m) = 0 \quad \rightarrow \quad m^2 - 5m + 6 = 0 \quad \rightarrow \quad m_1 = 2\,, \quad m_2 = 3$$

$$\therefore \quad y_c = c_1e^{2x} + c_2e^{3x}.$$

We have two linearly independent solutions e^{2x} and e^{3x} for the homogeneous eq. $(D^2 - 5D + 6)y = 0$.

Let

$$\left.\begin{array}{c} y = v\,e^{2x} \\ y' = 2v\,e^{2x} + v'\,e^{2x} \\ y'' = 4v\,e^{2x} + 4v'\,e^{2x} + v''\,e^{2x} \end{array}\right] \qquad (2)$$

Put (2) in (1), we get

$$4v\,e^{2x} + 4v'\,e^{2x} + v''\,e^{2x} - 10v\,e^{2x} - 5v'\,e^{2x} + 6v\,e^{2x} = 2e^{x}$$

$$e^{2x}\,v'' - e^{2x}\,v' = 2e^{x} \qquad \rightarrow \qquad v'' - v' = 2e^{-x} \qquad (3)$$

Let $w = v' \rightarrow w' = v''$ then eq.(3) becomes

$$w' - w = 2e^{-x} \quad \text{or} \quad \frac{dw}{dx} - w = 2e^{-x} \qquad (4)$$

eq. (4) is the first-order linear DE with $P(x) = -1$ and $Q(x) = 2e^{-x}$.

$$u(x) = e^{\int P(x)dx} = e^{-\int dx} = e^{-x}.$$

∴ The G.S. of eq. (4) is

$$w\,u(x) = \int u(x)\,Q(x)\,dx$$

$$we^{-x} = \int 2e^{-x}.e^{-x}\,dx = \int 2e^{-2x}dx = -e^{-2x} + c.$$

$$\therefore \quad w = -e^{-x} + ce^{x}.$$

But $w = v' = \frac{dv}{dx}$

$$\therefore \quad \frac{dv}{dx} = -e^{-x} + ce^{x} \quad \rightarrow \quad dv = (-e^{-x} + ce^{x})dx \quad \rightarrow \quad v = \int (-e^{-x} + ce^{x})dx$$

$\therefore \quad v = e^{-x} + ce^{x} + c_2 = e^{-x} + c_1e^{x} + c_2$, where $c_1 = c$.

The G.S. of eq. (1) is

$$y = v\,e^{2x} = (e^{-x} + c_1e^{x} + c_2)e^{2x} \qquad \rightarrow \qquad \boxed{y = e^{x} + c_1e^{3x} + c_2e^{2x}.}$$

Example: (non-constant coefficients)

Verify that $y = x$ is a particular solution of the equation:

$$2x^2y'' + xy' - y = 0 \qquad (1)$$

Find the G.S. of eq. (1). For what values of x the solution is valid.

Solution: $y = x \quad \rightarrow \quad y' = 1, \text{ and } y'' = 0$ (2)

From (2) and (1), we have

$$2x^2y'' + xy' - y = 2x^2.(0) + x\,(1) - x = x - x = 0.$$

$\therefore$ $y = x$ is a P.S. of eq. (1).

Let $\left.\begin{array}{c} y = vx \\ y' = v + v'x \\ y'' = 2v' + v''x \end{array}\right]$ (3)

Put (3) in (1), we get $2x^2(2v' + v''x) + x(v + v'x) - vx = 0$

$$4x^2v' + 2x^3v'' + xv + x^2v' - vx = 0 \quad \rightarrow \quad 2x^3v'' + 5x^2v' = 0$$

$$2xv'' + 5v' = 0 \qquad (4)$$

Let $w = v' \rightarrow w' = v''$ then eq. (4) becomes

$$2xw' + 5w = 0 \quad \rightarrow \quad 2x\frac{dw}{dx} = -5w \quad \rightarrow \quad \frac{dw}{w} = \frac{-5}{2x}dx$$

$$\rightarrow \int \frac{dw}{w} = \frac{-5}{2}\int \frac{dx}{x} \quad \rightarrow \quad \ln(w) = \frac{-5}{2}\ln(x) + \ln(k) = \ln\left(k\,x^{\frac{-5}{2}}\right)$$

$\rightarrow \quad w = k\,x^{\frac{-5}{2}}$, but $v' = \frac{dv}{dx} = w$

$$\therefore \quad \frac{dv}{dx} = k\,x^{\frac{-5}{2}} \quad \rightarrow \quad dv = k\,x^{\frac{-5}{2}}\,dx$$

$$\rightarrow \quad v = k\int x^{\frac{-5}{2}}\,dx \quad \rightarrow \quad v = \frac{-2k}{3}\,x^{\frac{-3}{2}} + c_1$$

$$= c_1 + c_2\,x^{\frac{-3}{2}} \text{ , where } c_2 = \frac{-2k}{3}.$$

The G.S. of eq. (1) is

$$y = vx = \left(c_1 + c_2\,x^{\frac{-3}{2}}\right)x = c_1x + c_2\,x^{\frac{-1}{2}} = c_1x + \frac{c_2}{\sqrt{x}}.$$

The solution is valid for all $x > 0$.

Example: (non-constant coefficients)

If $y_1 = e^x$ is the solution of $xy'' - 2(x+1)y' + (x+2)y = 0$. Find the G.S. of

$$xy'' - 2(x+1)y' + (x+2)y = xe^x \qquad (1)$$

Solution: Let $y = y_1\, v = e^x\, v$

$\rightarrow \quad y' = e^x\, v' + v\, e^x\,, \qquad y'' = e^x\, v'' + v'\, e^x + v\, e^x + e^x\, v' = \; e^x\, v'' + 2v'\, e^x + v\, e^x$

Put y, y', y'' in (1)

$$\rightarrow \quad x(e^x\, v'' + 2v'\, e^x + v\, e^x\,) + (-2x - 2)(e^x\, v' + v\, e^x) + (x + 2)e^x\, v$$

$$= x\, e^x\, v'' + 2x\, v' e^x + x v e^x - 2x e^x v' - 2x v e^x - 2e^x v' - 2v e^x + x e^x v + 2e^x v = x e^x.$$

$$\rightarrow \qquad x e^x v'' - 2e^x v' = x e^x\,\,] \div e^x$$

$$x\, v'' - 2v' = x$$

Let $w = v' \quad \rightarrow \quad w' = v''$

$$\rightarrow \quad xw' - 2w = x\,\,] \div x$$

$$\frac{dw}{dx} - \frac{2w}{x} = 1$$

$$u = e^{2\int \frac{dx}{x}} = e^{-2\ln x} = x^{-2}$$

$$\therefore \quad w\, x^{-2} = \int x^{-2}\, dx \quad \rightarrow \quad w x^{-2} = -x^{-1} + c\,] * x^2$$

$$\therefore \quad w = -x + cx^2$$

$$\therefore \quad v = \int (-x + cx^2) dx \quad \rightarrow \quad v = \frac{-x^2}{2} + c\frac{x^3}{3} + c_2$$

$$\boxed{y = ve^x = e^x\left(\frac{-x^2}{2} + c\frac{x^3}{3} + c_2\right)}$$

Exercises: Solve the following second-order equations by reduction of order method.

(1) $(D^2 - 3D + 2)y = \sqrt{1 + e^{2x}}$.
(2) $(D^2 + 4)y = sinx$.
(3) $(D^2 + 1)y = sec^3 x$. Use $y = v\, sinx$.

The Effect of the Operators:

I: We shall consider the effect of the operator D^n $(n=1,2,\dots)$ on e^{mx} where m is constant

$$De^{mx} = \frac{d}{dx}e^{mx} = me^{mx},\quad D^2e^{mx} = \frac{d^2}{dx^2}e^{mx} = m^2e^{mx},\ \text{and so on...}$$

$$\boxed{\text{In general } D^n e^{mx} = m^n e^{mx},\quad n=1,2,\dots} \qquad (1)$$

II: We shall consider the effect of the operator $(D-m)^n$, $(n=1,2,3,\dots)$ on ye^{mx}.

$$(D-m)ye^{mx} = \left(\frac{d}{dx}-m\right)ye^{mx} = \frac{d}{dx}(ye^{mx}) - mye^{mx}$$

(تفاضل ضرب حالتين)

$$= mye^{mx} + e^{mx}\frac{dy}{dx} - mye^{mx}$$

$$= e^{mx}\frac{dy}{dx} = e^{mx}Dy$$

$$(D-m)^2 ye^{mx} = (D-m)(D-m)ye^{mx} = (D-m)e^{mx}Dy$$

$$= D(e^{mx}Dy) - me^{mx}Dy$$

$$= e^{mx}D^2y + me^{mx}Dy - me^{mx}Dy = e^{mx}D^2y$$

and so on...

$$\boxed{\text{In general } (D-m)^n ye^{mx} = e^{mx}D^n y,\quad n=1,2,\dots} \qquad (2)$$

Now, Let $f(D)$ be a polynomial in D, that is

$$\boxed{f(D) = b_0D^n + b_1D^{n-1} + \dots + b_{n-1}D + b_n}$$

(i) consider the effect of the opertor $f(D)$ on e^{mx}

$$f(D)\, e^{mx} = (b_0 D^n + b_1 D^{n-1} + \cdots + b_{n-1} D + b_n) e^{mx}$$

$$= b_0 D^n e^{mx} + b_1 D^{n-1} e^{mx} + \cdots + b_{n-1} D e^{mx} + b_n e^{mx}.$$

Use of eq. (1), we have

$$f(D) e^{mx} = b_0 m^n e^{mx} + b_1 m^{n-1} e^{mx} + \cdots + b_{n-1} m e^{mx} + b_n e^{mx}$$

$$= e^{mx} (b_0 m^n + b_1 m^{n-1} + \cdots + b_{n-1} m + b_n)$$

$$\therefore \boxed{f(D) e^{mx} = e^{mx} f(m)} \quad \text{——— (3)}$$

If m is a root of the eq. $f(m) = 0$, then eq. (3) becomes

$$\boxed{f(D) e^{mx} = 0} \quad \text{——— (4)}$$

(ii) Consider the effect of the operator $f(D-m)$ on ye^{mx}

$$f(D-m)\, y e^{mx} = [b_0 (D-m)^n + b_1 (D-m)^{n-1} + \cdots + b_{n-1}(D-m) + b_n] y e^{mx}$$

$$= b_0 (D-m)^n y e^{mx} + b_1 (D-m)^{n-1} y e^{mx} + \cdots + b_{n-1}(D-m) y e^{mx} + b_n y e^{mx}.$$

Use of eq. (2), we have

$$f(D-m)\, y e^{mx} = b_0 e^{mx} D^n y + b_1 e^{mx} D^{n-1} y + \cdots + b_{n-1} e^{mx} D y + b_n e^{mx} y$$

$$= e^{mx} (b_0 D^n y + b_1 D^{n-1} y + \cdots + b_{n-1} D y + b_n y)$$

$$= e^{mx}\left(b_0 D^n + b_1 D^{n-1} + \cdots + b_{n-1} D + b_n\right) y$$

$$= e^{mx} f(D)\, y$$

$$\therefore \boxed{e^{mx} f(D)\, y = f(D-m)\, y\, e^{mx}} \quad \text{——(5)}$$

Eq.(5) shows (for differential operators with constant coefficients) how to shift an exponential factor from the left of a differential operator to the right of the differential operator.

(iii) $\boxed{(D-m)^n (x^k e^{mx}) = 0, \quad \text{for } k = 0, 1, 2, \cdots, n-1}$ ——(6)

Proof: Use of eq. (2) with $y = x^k$, we have

$$(D-m)^n (x^k e^{mx}) = e^{mx} D^n x^k.$$

But $D^n x^k = \frac{d^n}{dx^n} x^k = 0$, for $k = 1, 2, \cdots, n-1$

$$\therefore (D-m)^n (x^k e^{mx}) = 0, \quad \forall\, k = 1, 2, \cdots, n-1.$$

Note: Equations (3), (5), and (6) are very important in finding the G.S. of linear differential equations with constant coefficients.

Example: Find solutions for the following linear homogeneous differential equation

$$2y'' + 5y' - 12y = 0.$$

Show these solutions ~~solutions~~ are linearly independent, then find the general solution for the given equation.

Solution: $2y'' + 5y' - 12y = 0 \Rightarrow 2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 12y = 0$

$$\Rightarrow 2D^2y + 5Dy - 12y = 0 \Rightarrow (2D^2 + 5D - 12)\, y = 0.$$

Compare with eq-(3), we have

$f(D) = 2D^2+5D-12$ and the solution $y = e^{mx}$ where

m is the root of the equation $f(m) = 0$.

so $f(m) = 0 \Rightarrow 2m^2 + 5m - 12 = 0$

$(m+4)(2m-3) = 0$

$\therefore$ the roots are $m_1 = -4$ and $m_2 = \frac{3}{2}$

and $y_1 = e^{-4x}$ and $y_2 = e^{\frac{3}{2}x}$ are solutions for the equation

$(2D^2+5D-12)y = 0$, because

$(2D^2+5D-12)e^{-4x} = 0$ and $(2D^2+5D-12)e^{\frac{3}{2}x} = 0$.

Now, to check the linear independence of the solutions.

$$\begin{vmatrix} e^{-4x} & e^{\frac{3}{2}x} \\ -4e^{-4x} & \frac{3}{2}e^{\frac{3}{2}x} \end{vmatrix} = \frac{3}{2}e^{-4x}\cdot e^{\frac{3}{2}x} + 4e^{-4x}\cdot e^{\frac{3}{2}x}$$

$$= e^{-\frac{5}{2}x}\left(\frac{3}{2}+4\right) = \frac{11}{2}e^{-\frac{5}{2}x} \neq 0$$

$\therefore$ The solutions ~~are~~ $y_1 = e^{-4x}$ and $y_2 = e^{\frac{3}{2}x}$ are linearly independent and according to Theorem (1), the G.S. is

$$y = c_1y_1 + c_2y_2 = c_1e^{-4x} + c_2e^{\frac{3}{2}x}.$$

Example: Use the exponential shift and find the solutions of the equation $(D+3)^4 y = 0$ ——(*)

and show that the solutions are linearly independent.

Solution: Multiply the eq.(*) by $e^{3x} \Rightarrow e^{3x}(D+3)^4 y = 0$

Apply the exponential shift as given in eq.(5), where

$f(D) = (D+3)^4$ and $m = 3$

$$e^{3x}(D+3)^4 y = [(D+3)-3]^4 (ye^{3x}) = D^4(ye^{3x})$$

$$\therefore D^4(ye^{3x}) = 0 \Rightarrow ye^{3x} = c_1 + c_2 x + c_3 x^2 + c_4 x^3$$

$$\therefore \text{The G.S. is } y = c_1 e^{-3x} + c_2 x e^{-3x} + c_3 x^2 e^{-3x} + c_4 x^3 e^{-3x}$$

and the functions $y_1 = e^{-3x}$, $y_2 = xe^{-3x}$, $y_3 = x^2 e^{-3x}$, $y_4 = x^3 e^{-3x}$

are solutions for eq.(*).

Since the functions $1, x, x^2, x^3$ are linearly independent because

$$W = \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 2 & 6x \\ 0 & 0 & 0 & 6 \end{vmatrix} = (1)(1)(2)(6) = 12$$

and since $e^{-3x}, xe^{-3x}, x^2 e^{-3x}, x^3 e^{-3x}$ are functions of $1, x, x^2, x^3$

so the solutions $y_1 = e^{-3x}$, $y_2 = xe^{-3x}$, $y_3 = x^2 e^{-3x}$, $y_4 = x^3 e^{-3x}$

are linearly independent.

Exercise to prove: Show that the functions $e^{mx}, xe^{mx}, x^2 e^{mx}, \ldots, x^{n-1} e^{mx}$ are linearly independent.

Exercises to solve:

Use the exponential shift and find the G.S. of the following linear differential equations

1. $(D-2)^3 y = 0$
2. $(D+1)^2 y = 0$
3. $(2D+1)^2 y = 0$
4. $(D+7)^6 y = 0$

Inverse Differential Operators:

The Operator $\frac{1}{f(D)}$:

When we require a particular solution y_p of the equation

$$f(D)y = R(x). \quad \text{---} (1)$$

It is natural to write $\boxed{y = \frac{1}{f(D)} R(x)}$ ---- (2)

where $\frac{1}{f(D)}$ is called the inverse differential operator.

Evaluation of $\frac{1}{f(D)} e^{ax}$:

From chapter 3, we have

$$f(D)e^{ax} = e^{ax} f(a) \quad \text{---} (3)$$

and $$(D-a)^n (x^n e^{ax}) = n!\, e^{ax} \quad \text{---} (4)$$

Now, eq.(3) suggests

$$\boxed{\frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)}}, \quad f(a) \neq 0. \quad \text{---} (5)$$

Example (1): Solve the eq. $(D^2+1)y = e^{2x}$ by using inverse differential operator for finding a particular solution y_p.

Solution: $f(D)y = e^{2x}$, $(D^2+1)y = e^{2x}$

$f(m) = 0 \Rightarrow m^2 + 1 = 0 \Rightarrow m = \pm i \Rightarrow y_c = c_1 \cos x + c_2 \sin x$.

$$y_p \overset{\text{eq.(2)}}{=} \frac{1}{D^2+1} e^{2x} \overset{\text{eq.(5)}}{=} \frac{e^{2x}}{f(2)} = \frac{1}{5} e^{2x}.$$

$f(D) = D^2 + 1$, $a = 2$. $\Rightarrow f(2) = 2^2 + 1 = 5 \neq 0$. $\therefore f(a) \neq 0$

$\therefore$ The G.S. is $\boxed{y = y_c + y_p = c_1 \cos x + c_2 \sin x + \frac{1}{5} e^{2x}}$.

Now in eq.(5), if $f(a) = 0$, then $f(D)$ contain the factor $(D-a)$. Suppose that this factor occurs precisely n times in $f(D)$, that is $f(D) = \phi(D)(D-a)^n$, $\phi(a) \neq 0$.

By using eq.(4), we have

$$\boxed{\frac{1}{\phi(D)(D-a)^n} e^{ax} = \frac{x^n e^{ax}}{n! \phi(a)}} \quad \text{---------- (6)}$$

$\phi(D)(D-a)^n = f(D)$

Example(12): Solve the equation $D^2(D-1)^3(D+1)y = e^x$ by using inverse differential operator for finding y_p.

Solution: $D^2(D-1)^3(D+1)y = e^x$

$f(m) = 0 \Rightarrow m^2(m-1)^3(m+1) = 0 \Rightarrow m = 0, 0, 1, 1, 1, -1$

$\therefore y_c = c_1 + c_2 x + (c_3 + c_4 x + c_5 x^2) e^x + c_6 e^{-x}$.

$f(D) = D^2(D-1)^3(D+1)$, $a = 1$, $n = 3$

$f(a) = f(1) = 0$. (So we have to use eq. (6)).

We write $f(D) = \phi(D)(D-a)^n = \overbrace{D^2(D+1)}^{\phi(D)}(D-1)^{3^n}$ where

$\phi(D) = D^2(D+1) \Rightarrow \phi(a) = \phi(1) = 1^2(1+1) = 2$.

$\therefore y_p = \frac{1}{D^2(D+1)(D-1)^3} e^x = \frac{x^3 e^x}{3!(2)} = \frac{1}{12} x^3 e^x$.

$\therefore$ The G-S. is $y = y_c + y_p = c_1 + c_2 x + (c_3 + c_4 x + c_5 x^2)e^x + c_6 e^{-x} + \frac{1}{12} x^3 e^x$.

EX(3): Solve the equation $(D^2+3D-4)y = 20e^{-4x}$.

Solution: $f(D) = D^2+3D-4$.

$f(m) = 0 \Rightarrow m^2+3m-4 = 0 \Rightarrow (m-1)(m+4) = 0 \Rightarrow m = 1, -4$.

$\therefore y_c = c_1 e^x + c_2 e^{-4x}$.

$f(D) = D^2+3D-4$, $a = -4 \Rightarrow f(a) = 0$

we write $f(D) = \phi(D)(D-a)^n = (D-1)(D+4)$

where $\phi(D) = (D-1)$.

$\therefore y_p = \frac{1}{D^2+3D-4} 20e^{-4x} = 20 \frac{1}{(D-1)(D+4)} e^{-4x} = 20 \frac{x e^{-4x}}{1!(-4-1)}$

$= -4x e^{-4x}$.

$\therefore$ The G-S. is $y = y_c + y_p$

$\therefore \boxed{y = c_1 e^x + c_2 e^{-4x} - 4x e^{-4x}}$

Exercises: Use inverse differential operator for finding the p.s. of y_p.

(1) $(D^2 - 3D + 2) = e^{2x}$

(2) $(4D^2 - 1)y = e^{\frac{x}{2}} + 12e^x$.

Ex(4): Solve the eq. $(D^2 - 3D + 2)y = e^x + e^{2x}$.

Solution: $f(m) = 0 \Rightarrow m^2 - 3m + 2 = 0 \Rightarrow m = 1, 2$.

$\therefore y_c = c_1 e^x + c_2 e^{2x}$.

$$y_p = \frac{1}{D^2 - 3D + 2}(e^x + e^{2x}) = \frac{1}{(D-2)(D-1)} e^x + \frac{1}{(D-1)(D-2)} e^{2x}$$

$$= \frac{x e^x}{1!\,(1-2)} + \frac{x e^{2x}}{1!\,(2-1)} = -x e^x + x e^{2x}$$

$$= x(e^{2x} - e^x).$$

$\therefore$ The G.S. is $y = y_c + y_p$

$$\therefore y = c_1 e^x + c_2 e^{2x} + x(e^{2x} - e^x).$$

Chapter 5

Laplace Transform

Laplace Transform: The Laplace transform of the function $f(t)$, denoted by $L\{f(t)\}$ is defined by

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s) \qquad (1)$$

Where the integral of eq. (1) is a function of the parameter s.

Remark: The operator L, like the differential operator D, is a linear operator. That is, if $f_1(t), f_2(t), \cdots, f_n(t)$ have Laplace transforms and $c_1, c_2, \cdots, c_n$ are constants, then

$$L\{c_1 f_1(t) + c_2 f_2(t) + \cdots + c_n f_n(t)\} = c_1 L\{f_1(t)\} + c_2 L\{f_2(t)\} + \cdots + c_n L\{f_n(t)\}.$$

Example.1: Find the Laplace transform of the function

$$f(t) = t\,, \quad 0 < t < 4$$
$$= 5\,, \quad t > 4.$$

Solution:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \int_0^4 t\ e^{-st}\,dt + \int_4^\infty 5\,e^{-st}\ dt$$

$$L\{f(t)\} = \left[\frac{-t}{s}\,e^{-st}\Big|_0^4 - \frac{1}{s^2}e^{-st}\Big|_0^4\right] + \left[\frac{-5}{s}\,e^{-st}\Big|_4^\infty\right]$$

$$L\{f(t)\} = \left[\frac{-t}{s}\,e^{-st} - \frac{1}{s^2}e^{-st}\right]\Big|_0^4 + \left[\frac{-5}{s}\,e^{-st}\Big|_4^\infty\right]$$

$$= \left[\left(\frac{-4}{s}\,e^{-4s} - \frac{1}{s^2}\,e^{-4s}\right) - \left(0 - \frac{1}{s^2}\,e^{0}\right)\right] + \left[\frac{-5}{s}(e^{-\infty} - e^{-4s})\right]$$

$$= \left[\frac{-4}{s}e^{-4s} - \frac{1}{s^2}e^{-4s} + \frac{1}{s^2}\right] + \frac{5}{s}\,e^{-4s} = \frac{1}{s^2} + \frac{1}{s}\,e^{-4s} - \frac{1}{s^2}\,e^{-4s}.$$

Transforms of Elementary Functions:

(1) Evaluation of $L\{e^{kt}\}$, k is constant

For $s > k$,

$$L\{e^{kt}\} = \int_0^\infty e^{-st}\,e^{kt}\ dt = \int_0^\infty e^{-(s-k)t}\ dt = -\frac{1}{s-k}\,e^{-(s-k)t}\Big|_0^\infty$$

$$= -\frac{1}{s-k}(e^{-\infty} - e^{0}) = -\frac{1}{s-k}(0-1) = \frac{1}{s-k}.$$

If $k = 0$, then $L\{1\} = \frac{1}{s}$ and $L\{k\} = k\,L\{1\} = \frac{k}{s}$.

(2) Evaluation of $L\{sinkt\}$ and $L\{coskt\}$

$$L\{sinkt\} = L\left\{\frac{e^{ikt} - e^{-ikt}}{2i}\right\} = \frac{1}{2i}\left[L\{e^{ikt}\} - L\{e^{-ikt}\}\right]$$

$$= \frac{1}{2i}\left(\frac{1}{s - ik} - \frac{1}{s + ik}\right) = \frac{1}{2i}\,\frac{(s + ik) - (s - ik)}{(s - ik)(s + ik)}$$

$$= \frac{1}{2i}\,\frac{s + ik - s + ik}{s^2 - i^2k^2} = \frac{1}{2i}\,\frac{2ik}{s^2 + k^2} = \frac{k}{s^2 + k^2}.$$

$$L\{coskt\} = L\left\{\frac{e^{ikt} + e^{-ikt}}{2}\right\} = \frac{1}{2}\left[L\{e^{ikt}\} + L\{e^{-ikt}\}\right]$$

$$= \frac{1}{2}\left(\frac{1}{s - ik} + \frac{1}{s + ik}\right) = \frac{1}{2}\,\frac{s + ik + s - ik}{(s - ik)(s + ik)} = \frac{1}{2}\,\frac{2s}{s^2 + k^2}.$$

(3) Evaluation of $L\{sinhkt\}$ and $L\{coshkt\}$

$$L\{sinhkt\} = L\left\{\frac{e^{kt} - e^{-kt}}{2}\right\} = \frac{1}{2}\left[L\{e^{kt}\} - L\{e^{-kt}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s - k} - \frac{1}{s + k}\right] = \frac{1}{2}\,\frac{s + k - s + k}{(s + k)(s - k)}$$

$$= \frac{1}{2}\,\frac{2k}{s^2 - k^2} = \frac{k}{s^2 - k^2}.$$

$$L\{coshkt\} = L\left\{\frac{e^{kt} + e^{-kt}}{2}\right\} = \frac{1}{2}\left[L\{e^{kt}\} + L\{e^{-kt}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s - k} + \frac{1}{s + k}\right] = \frac{1}{2}\,\frac{s + k + s - k}{(s + k)(s - k)}$$

$$= \frac{1}{2} \frac{2s}{s^2 - k^2} = \frac{s}{s^2 - k^2}.$$

(4) Evaluation of $L\{t^n\}$, n is positive integer

$$L\{t^n\} = \int_0^\infty t^n \, e^{-st} \, dt,$$

Integrate by parts, yields: $u = t^n$, $dv = e^{-st}dt$

$$du = n\, t^{n-1}\, dt, \qquad v = \frac{-1}{s}\, e^{-st}.$$

$$L\{t^n\} = \frac{-1}{s}\, t^n \, e^{-st}\Big|_0^\infty + \frac{n}{s} \int_0^\infty t^{n-1} \, e^{-st} \, dt = 0 + \frac{n}{s} \int_0^\infty t^{n-1} \, e^{-st} \, dt$$

$$L\{t^n\} = \frac{n}{s}\, L\{t^{n-1}\}$$

$$L\{t^{n-1}\} = \frac{n-1}{s}\, L\{t^{n-2}\}$$

So

$$L\{t^n\} = \frac{n}{s} \frac{n-1}{s}\, L\{t^{n-2}\} = \frac{n(n-1)}{s^2}\, L\{t^{n-2}\}$$

Iteration of this process, we have

$$L\{t^n\} = \frac{n(n-1)(n-2)\cdots[n-(n-1)]}{s^n}\, L\{t^{n-n}\}$$

$$= \frac{n(n-1)(n-2)\cdots 3.2.1}{s^n}\, L\{1\} = \frac{n!}{s^n} \cdot \frac{1}{s} = \frac{n!}{s^{n+1}}.$$

Example 2: Obtain $L\{2t^3 - 5t^2 + 4t + 12\}$.

Solution:

$$L\{2t^3 - 5t^2 + 4t + 12\} = 2L\{t^3\} - 5L\{t^2\} + 4L\{t\} + 12L\{1\}$$

$$= 2\frac{3!}{s^4} - 5\frac{2!}{s^3} + 4\frac{1!}{s^2} + 12\frac{1}{s} = \frac{12}{s^4} - \frac{10}{s^3} + \frac{4}{s^2} + \frac{12}{s}.$$

Example 3: Obtain $L\{2t^3 - 4e^{-5t} + 2sin3t\}$.

Solution:

$$L\{2t^3 - 4e^{-5t} + 2sin3t\} = 2L\{t^3\} - 4L\{e^{-5t}\} + 2L\{sin3t\}$$

$$= 2\frac{3!}{s^4} - 4\frac{1}{s+5} + 2\frac{3}{s^2+9} = \frac{12}{s^4} - \frac{4}{s+5} + \frac{6}{s^2+9}.$$

Example 4: Obtain $L\{2e^{3t} + sinh\,4t - 2cos7t\}$.

Solution:

$$L\{2e^{3t} + sinh\,4t - 2cos7t\} = 2L\{e^{3t}\} + L\{\sinh 4t\} - 2L\{\cos 7\,t\}$$

$$= 2\frac{1}{s-3} + \frac{4}{s^2-16} - 2\frac{s}{s^2+49} = \frac{2}{s-3} + \frac{4}{s-3} - \frac{2s}{s^2+49}.$$

Example 5: Find $L\{sin^2\,3x\}$.

Solution:

$$L\{sin^2\,3x\} = L\left\{\frac{1-cos6x}{2}\right\} = \frac{1}{2}\,[L\{1\} - L\{cos6x\}]$$

$$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+36}\right) = \frac{1}{2}\,\frac{s^2+36-s^2}{s(s^2+36)} = \frac{18}{s(s^2+36)}.$$

Example 6: Find $L\{sin3x\ \ cos3x\}$.

Solution:

$$L\{sin3x\ \ cos3x\} = L\left\{\frac{1}{2}\ sin6x\right\} = \frac{1}{2}\,L\{sin6x\} = \frac{1}{2}\,\frac{6}{s^2+36} = \frac{3}{s^2+36}.$$

Laplace Inverse:

Since $L\{f(t)\} = F(s) = \int_0^\infty e^{-st}\,f(t)\,dt$

Then $L^{-1}\{F(s)\} = f(t)$.

Example 1: Find $L^{-1}\left\{\frac{1}{s^3-s}\right\}$.

Solution:

$$\frac{1}{s^3-s} = \frac{1}{s(s^2-1)} = \frac{1}{s(s-1)(s+1)}$$

$$\frac{1}{s(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1} = \frac{A(s-1)(s+1) + B\,s(s+1) + C\,s(s-1)}{s(s-1)(s+1)}$$

$$\Rightarrow\ \ 1 = A(s-1)(s+1) + B\,s(s+1) + C\,s(s-1)$$

Let

$$s = 0 \;\Rightarrow\; 1 = -A \;\Rightarrow\; A = -1$$

$$s = 1 \;\Rightarrow\; 1 = 2B \;\Rightarrow\; B = \frac{1}{2}$$

$$s = -1 \;\Rightarrow\; 1 = 2C \;\Rightarrow\; C = \frac{1}{2}$$

$$\therefore \; \frac{1}{s(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1} = \frac{-1}{s} + \frac{\frac{1}{2}}{s-1} + \frac{\frac{1}{2}}{s+1} = \frac{-1}{s} + \frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\frac{1}{s+1}.$$

$$\therefore \; L^{-1}\left\{\frac{-1}{s} + \frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\frac{1}{s+1}\right\} = -1 + \frac{1}{2}e^{t} + \frac{1}{2}e^{-t}.$$

Theorem 1: $L^{-1}\{F(s-a)\} = e^{at}\,L^{-1}\{F(s)\}$.

Proof: Since $F(s-a) = \int_0^\infty e^{-(s-a)t}\,f(t)\,dt$

$$= \int_0^\infty e^{-st}\,e^{at}\,f(t)dt = L\{e^{at}\,f(t)\}.$$

Then $L^{-1}\{F(s-a)\} = e^{at}\,f(t) = \; e^{at}\,L^{-1}\{F(s)\}$.

Example 2: Find $L^{-1}\left\{\frac{15}{s^2+4s+13}\right\}$.

Solution: $L^{-1}\left\{\frac{15}{s^2+4s+13}\right\} = \; L^{-1}\left\{\frac{15}{s^2+4s+4+9}\right\} = L^{-1}\left\{\frac{15}{(s+2)^2+9}\right\}$

Using Theorem 1,

$$L^{-1}\left\{\frac{15}{(s+2)^2+9}\right\} = e^{-2t}\,L^{-1}\left\{\frac{15}{s^2+9}\right\}$$

$$= 5\;e^{-2t}\,L^{-1}\left\{\frac{3}{s^2+9}\right\} = 5\,e^{-2t}\,sin3t.$$

Example 3: Find $L^{-1}\left\{\frac{s+1}{s^2+6s+25}\right\}$.

Solution: $L^{-1}\left\{\frac{s+1}{s^2+6s+25}\right\} = L^{-1}\left\{\frac{s+1}{s^2+6s+9+16}\right\} = L^{-1}\left\{\frac{s+3-2}{s^2+6s+9+16}\right\} = L^{-1}\left\{\frac{(s+3)-2}{(s+3)^2+16}\right\}$.

Then by Theorem 1,

$$L^{-1}\left\{\frac{(s+3)-2}{(s+3)^2+16}\right\} = e^{-3t}\, L^{-1}\left\{\frac{s-2}{s^2+16}\right\}$$

$$= e^{-3t}\left[L^{-1}\left\{\frac{s}{s^2+16}\right\} - L^{-1}\left\{\frac{2}{s^2+16}\right\}\right]$$

$$= e^{-3t}\left[\cos 4t - \frac{1}{2}\, sin4t\right].$$

Exercises: Find

(1) $L^{-1}\left\{\frac{1}{s^2+2s+10}\right\}$

(2) $L^{-1}\left\{\frac{1}{s^2+4s+4}\right\}$

(3) $L^{-1}\left\{\frac{3s}{s^2+4s+13}\right\}$

(4) $L^{-1}\left\{\frac{2s-3}{s^2-4s+4}\right\}$

Solution of (4):

$$L^{-1}\left\{\frac{2s-3}{s^2-4s+4}\right\} = L^{-1}\left\{\frac{2s-4+1}{(s-2)^2}\right\} = L^{-1}\left\{\frac{2(s-2)+1}{(s-2)^2}\right\}.$$

Apply Theorem 1, yields

$$L^{-1}\left\{\frac{2(s-2)+1}{(s-2)^2}\right\} = e^{2t}\, L^{-1}\left\{\frac{2s+1}{s^2}\right\}$$

$$= e^{2t}\left[L^{-1}\left\{\frac{2}{s}\right\} + L^{-1}\left\{\frac{1}{s^2}\right\}\right] = e^{2t}\,[2+t].$$

Laplace Transform of Derivatives:

In order to use Laplace transform in solving differential equation, I.V.P. , first learn how to transform the derivatives.

Let

$$y = y(t), \qquad \frac{dy}{dt} = y'(t), \qquad \frac{d^2y}{dt^2} = y''(t), \qquad \frac{d^3y}{dt^3} = y'''(t), \cdots$$

have Laplace transform then

$$L\{y(t)\} = \int_0^\infty e^{-st}\ y(t)\ dt$$

$$L\{y'(t)\} = L\left\{\frac{dy}{dt}\right\} = \int_0^\infty e^{-st}\ \frac{dy}{dt}\ dt$$

integrate by parts

$$u = e^{-st}, \qquad dv = \frac{dy}{dt}\ dt$$
$$du = -s\ e^{-st}, \qquad v = y(t)$$

$$\therefore\ L\left\{\frac{dy}{dt}\right\} = y(t)\ e^{-st}|_0^\infty + s\int_0^\infty e^{-st}\ y(t)\ dt = 0 - y(0) + s\ L\{y(t)\}$$

$$\therefore\ \boxed{L\left\{\frac{dy}{dt}\right\} = s\ L\{y(t)\} - y(0).}$$

Also,

$$L\left\{\frac{d^2y}{dt^2}\right\} = L\left\{\frac{dy'}{dt}\right\} = sL\{y'(t)\} - y'(0) = s[sL\{y(t)\} - y(0)] - y'(0)$$

$$\therefore\ \boxed{L\left\{\frac{d^2y}{dt^2}\right\} = s^2\ L\{y(t)\} - sy(0) - y'(0)}$$

Similarly

$$L\left\{\frac{d^3y}{dt^3}\right\} = L\left\{\frac{dy''}{dt}\right\} = sL\{y''(t)\} - y''(0) = s[s^2L\{y(t)\} - sy(0) - y'(0)] - y''(0)$$

$$\therefore \boxed{L\left\{\frac{d^3y}{dt^3}\right\} = s^3\,L\{y(t)\} - s^2\,y(0) - sy'(0) - y''(0).}$$

In general,

$$\boxed{L\left\{\frac{d^ny}{dt^n}\right\} = s^n\,L\{y(t)\} - s^{n-1}\,y(0) - \cdots - sy^{n-2}(0) - y^{n-1}(0).}$$

For example, if $n = 5$

$$\Rightarrow \quad L\left\{\frac{d^5y}{dt^5}\right\} = s^5\,L\{y(t)\} - s^4\,y(0) - s^3y'(0) - s^2y''(0) - s\,y'''(0) - y''''(0).$$

Usage of Laplace Transform in Solving DE's. (I.V.P.):

After learning how to compute the Laplace transform $F(s)$ of a function $f(t)$ and how to compute the Laplace form of derivatives, we will see in this section that the Laplace transform converts a differential equation in the unknown function $y(t)$ into an algebraic equation in $Y(s)$. Because algebraic equations are generally easier to solve than differential equations, this is one method that simplifies the problem of finding the solution $y(t)$.

Example 1: Solve the initial value problem

$$\frac{dy}{dt} = e^{2t}, \qquad y(0) = \frac{1}{2}. \qquad (1)$$

Solution: First take Laplace transform to both sides of eq. (1)

$$\Rightarrow \quad L\left\{\frac{dy}{dt}\right\} = L\{e^{2t}\} = \frac{1}{s-2},$$

but $L\left\{\frac{dy}{dt}\right\} = s\,L\{y(t)\} - y(0)$

$$\therefore \quad s\,L\{y(t)\} - y(0) = \frac{1}{s-2}$$

$$\Rightarrow \; s\,L\{y(t)\} - \frac{1}{2} = \frac{1}{s-2} \quad \Rightarrow \quad s\,L\{y(t)\} = \frac{1}{s-2} + \frac{1}{2}$$

$$= \frac{2+s-2}{2(s-2)} = \frac{s}{2(s-2)}.$$

$$\therefore \quad Y(s) = L\{y(t)\} = \frac{1}{2}\ \frac{1}{s-2},$$

Then using Laplace inverse,

$$\Rightarrow \quad y(t) = \frac{1}{2}\ e^{2t}.$$

Example 2: Solve the IVP $y'' + y = 0$, $y(0) = 0,\ y'(0) = 1$.

Solution: $L\{y'' + y\} = 0 \quad \Rightarrow \quad L\{y''\} + L\{y\} = 0$

$$[s^2\, L\{y\} - s\, y(0) - y'(0)] + L\{y\} = 0$$

$$(s^2+1)\, L\{y\} = s\, y(0) + y'(0) \quad \Rightarrow \quad (s^2+1)\, L\{y\} = 0 + 1 \quad \Rightarrow \quad L\{y\} = Y(s) = \frac{1}{s^2+1}$$

$$\therefore \quad y(t) = sint.$$

Example 3: Solve the IVP $y'' + y = 1$, $y(0) = 2,\ y'(0) = 0$.

Solution: $L\{y'' + y\} = L\{1\} \quad \Rightarrow \quad L\{y''\} + L\{y\} = \frac{1}{s}$

$$\Rightarrow \quad [s^2\, L\{y\} - s\, y(0) - y'(0)] + L\{y\} = \frac{1}{s}$$

$$(s^2+1)\, L\{y\} = \frac{1}{s} + s\, y(0) + y'(0) = \frac{1}{s} + 2s + 0$$

$$\Rightarrow \quad (s^2+1)\, L\{y\} = \frac{1}{s} + 2s \quad \Rightarrow \quad (s^2+1)\ L\{y\} = \frac{1+2s^2}{s}$$

$$\Rightarrow \quad L\{y\} = \frac{2s^2+1}{s\,(s^2+1)} = \frac{s^2+s^2+1}{s\,(s^2+1)}$$

$$\therefore \quad L\{y(t)\} = \frac{1}{s} + \frac{s}{s^2+1}$$

$$\therefore \quad y(t) = L^{-1}\left\{\frac{1}{s}\right\} + L^{-1}\left\{\frac{s}{s^2+1}\right\} = 1 + cost.$$

$$\therefore \quad y(t) = 1 + cost.$$

Example 4: Solve $y'' + 4y' + 8y = sint\,,\quad y(0) = 1\,,\quad y'(0) = 0\,.$

Solution: $L\{y'' + 4y' + 8y\} = L\{sint\}$

$$L\{y''\} + 4\,L\{y'\} + 8\,L\{y\} = \frac{1}{s^2+1}$$

$$[s^2\,Y(s) - sy(0) - y'(0)] + 4\,[s\,Y(s) - y(0)] + 8Y(s) = \frac{1}{s^2+1}$$

$$[s^2\,Y(s) - s - 0] + 4\,[sY(s) - 1] + 8\,Y(s) = \frac{1}{s^2+1}$$

$$[s^2 + 4s + 8]\,Y(s) = \frac{1}{s^2+1} + (s+4)$$

$$\therefore \quad Y(s) = \frac{s+4}{s^2+4s+8} + \frac{1}{(s^2+1)+(s^2+4s+8)}\,.$$

Then

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{s+4}{s^2+4s+8}\right\} + L^{-1}\left\{\frac{1}{(s^2+1)(s^2+4s+8)}\right\}$$

$$\Rightarrow \quad L^{-1}\left\{\frac{s+4}{s^2+4s+8}\right\} + \left[L^{-1}\left\{\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4s+8}\right\}\right]$$

Then, apply Theorem 1,

$$\begin{aligned}
\Rightarrow \quad & L^{-1}\left\{\frac{(s+2)+2}{s^2+4s+4+4}\right\} + \left[L^{-1}\left\{\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4s+4+4}\right\}\right] \\
&= L^{-1}\left\{\frac{(s+2)+2}{(s+2)^2+4}\right\} + \left[L^{-1}\left\{\frac{As+B}{s^2+1} + \frac{Cs+D}{(s+2)^2+4}\right\}\right] \\
&= e^{-2t}L^{-1}\left\{\frac{s+2}{s^2+4}\right\} + \left[L^{-1}\left\{\frac{As+B}{s^2+1} + \frac{Cs+D}{(s+2)^2+4}\right\}\right] \\
&= e^{-2t}\left[L^{-1}\left\{\frac{s}{s^2+4}\right\} + L^{-1}\left\{\frac{2}{s^2+4}\right\}\right] + \left[L^{-1}\left\{\frac{As+B}{s^2+1} + \frac{Cs+D}{(s+2)^2+4}\right\}\right] \\
&= e^{-2t}[cos2t + sin2t] + \left[L^{-1}\left\{\frac{As+B}{s^2+1} + \frac{Cs+D}{(s+2)^2+4}\right\}\right]
\end{aligned}$$

$$\therefore \quad y(t) = [\mathbf{H}.\mathbf{W}.]$$

Examples (1) - (4) illustrate the solution procedure that is outlined in Figure I.

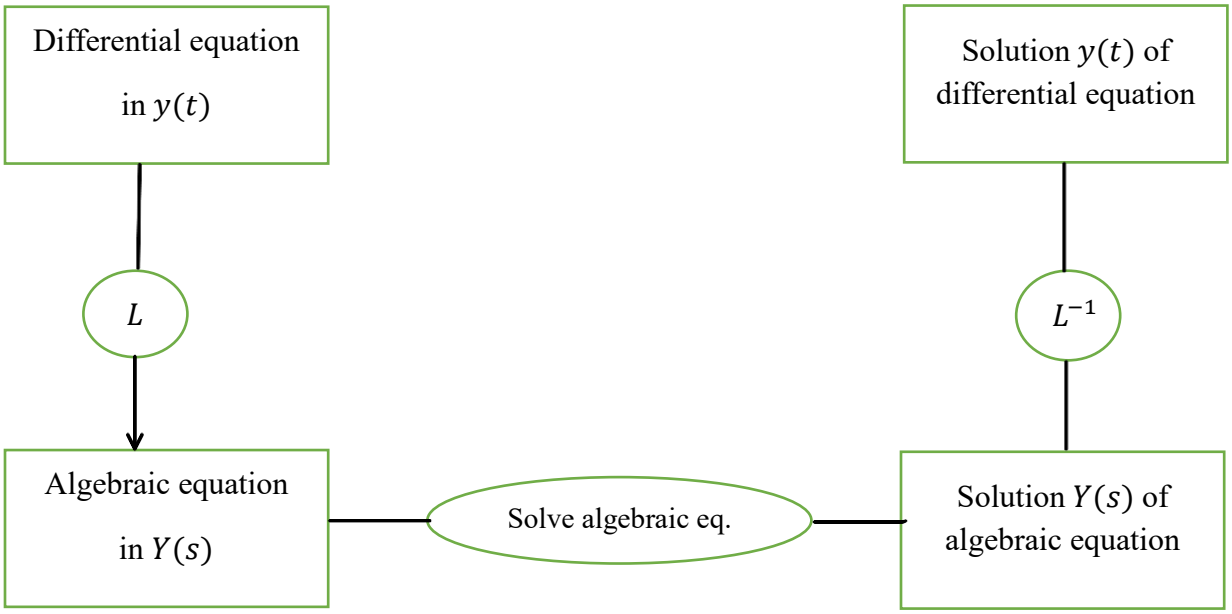

Figure I: Using Laplace transform to solve IVP.

Exercises: Solve the following IVP by using Laplace transform method.

(1) $y' + y = e^{2t}$, $y(0) = 0$
(2) $y' + 20y = 6\, sin2t$, $y(0) = 6$
(3) $y'' + 2y' - 3y = sin2t$, $y(0) = y'(0) = 0$
(4) $y'' - 3y' + 2y = e^{3t}$, $y(0) = y'(0) = 0$.